# THREE DIMENSIONAL GEOMETRY MAINTENANCE FOR FORMATION FLYING ON ELLIPTIC ORBITS

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## ABSTRACT

There has been impending interest in the formation flying with many satellites. Multiple satellite system enhances the missions' flexibility with less total mass and cost, and realizes some missions that were impossible with a single satellite. At the Institute of Space and Astronautical Science (ISAS/JAXA), the plasma and magnetic field observation missions with several satellites is under investigation. The mission under consideration is designated as SCOPE. The observation area of the SCOPE mission is twenty or thirty earth radii away from the center of the earth where the geomagnetic field has interaction with the energetic particles from the sun. Therefore its orbit becomes highly elliptic. This paper first discusses the design method for spontaneous maintaining the formation geometry on the elliptic orbits. In the observation aspect, the formation of plural satellites is requested to constitute a polygon that assures the high spatial resolution observation. This study next show the orbital design method for the SCOPE mission. The frozen property that maintains high spatial resolution near the apogee is found feasible for elliptic orbit. Numerical examples are presented with practical illustrations.

## NOMENCLATURE

| | | |
|---|---|---|
| $a$ | : | semi-major axis |
| $e$ | : | eccentricity |
| $i$ | : | inclination |
| $\Omega$ | : | right ascension of ascending node |
| $\omega$ | : | argument of perigee |
| $t_0$ | : | perigee passage time |
| $t$ | : | time |
| $f$ | : | true anomaly |
| $M$ | : | mean anomaly |
| $n$ | : | mean motion |
| $p$ | : | semi-latus rectum |
| $\mu$ | : | gravity constant |
| $\mathbf{r}$ | : | relative position |

## 1. INTRODUCTION

In recent years, there has been impending interest in the formation flying with many satellites. Multiple satellite system enhances the missions' flexibility with less total mass and cost, and realizes some missions that were impossible with a single satellite, for example, infrared ray or laser interferometers, multi point magnetic field observation, large space antenna and so on. At the Institute of Space and Astronautical Science (ISAS, JAXA), the plasma and magnetic field observation mission with multiple satellites is under consideration. The mission under consideration is designated as SCOPE (SCOPE: Scale COoupling in Plasma Environment), the successor mission to GEOTAIL. GEOTAIL has observed the plasma surrounding of the earth, but it cannot distinguish the time fluctuation with special distribution because it is a single satellite mission. To overcome this difficulty, in the SCOPE mission, the observation of many points using the plural satellites is required. In the observation aspect, the formation of plural satellites is requested to constitute a polygon that assures the high spatial resolution observation.

The observation area of the SCOPE mission is twenty or thirty earth radii away from the center of the earth where the geomagnetic field has interaction with the energetic particles from the sun. Therefore its orbit becomes highly elliptic. On such highly elliptic orbit, it is very difficult to keep the geometry of formation flying because the relative positions of the satellites change largely as time goes on. One possible solution to maintain the satellites' geometry is positive control of relative position with satellites' fuel. But it is unrealistic because the fuel consumptions of satellites probably become large. So the design of the orbit that spontaneously keeps the high spatial resolution is necessary. It is known that the three-dimensional formation maintenance over the orbit is impossible. But, as the focused area of SCOPE mission is near the apogee, the designing the orbit that keep the polygon becomes possible.

This paper first discusses the design method for spontaneous maintaining the formation geometry on the elliptic orbits. The relative position of satellites on the elliptic orbit can be expressed with the small differences of the Keplerian parameters. By giving the adequate values to these parameters, we can design many interesting orbits. The orbit which can maintains the distance between the satellites or two-dimensional geometry can be designed. And this study next show the orbital design method for the SCOPE mission. The frozen property that maintains high spatial resolution near the apogee is found feasible for elliptic orbit. Numerical examples are presented with practical illustrations.

## 2. RELATIVE POSITION ON ELLIPTIC ORBITS

First, the coordinate system is defined as Figure 1.

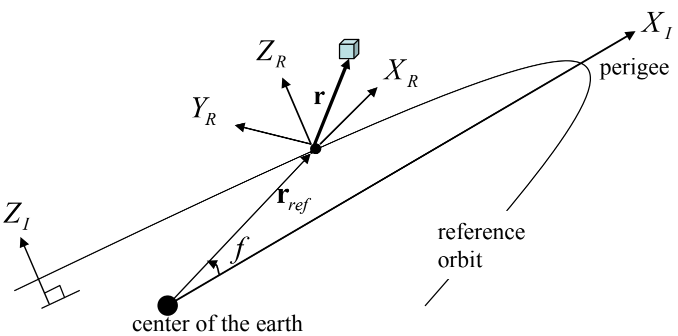


Figure 1: Reference orbit and rotating coordinate

The orbital parameters of the reference satellite on the reference orbit are $(a,e,i,\Omega=0,\omega=0,t_0)$. The rotating frame is considered here; $X_R$ axis is the radius vector and $Z_R$ axis points the angular velocity direction. The position vector of reference satellite is

$$\mathbf{r}_{ref} = (r,0,0)^T,\ r = \frac{p}{1+e\cos f}. \tag{1}$$

Here the satellite whose orbital parameters are $(a',e',i',\Omega',\omega',t_0')$ is considered. When this satellite is located near the reference satellite, these orbital parameters should be written as follows;

$$\begin{aligned} &a' = a+\delta a,\ e' = e+\delta e,\ i' = \delta i,\ t_0' = t_0+\delta t_0 \\ &\Omega'+\omega' = 2\pi\times k+\delta\omega\ \ (k=0,\pm1,\pm2,...) \end{aligned}, \tag{2}$$

where

$$|\delta a| << 1,\ |\delta e| << 1,\ |\delta i| << 1,\ |\delta\omega| << 1,\ |\delta t_0| << 1. \tag{3}$$

The position vector of this satellite in the rotating frame can be obtained.

$$\mathbf{r}_{sat} = R_Z(-f)R_Z(\Omega')R_X(\delta i)R_Z(\omega'+f')\begin{pmatrix} r' \\ 0 \\ 0 \end{pmatrix}. \tag{4}$$

$f'$ is true anomaly and $r'$ is radius of satellite respectively.

$$r' = r_{ref} + \left.\frac{\partial r_{ref}}{\partial a}\right|_f \delta a + \left.\frac{\partial r_{ref}}{\partial e}\right|_f \delta e + \frac{1}{n}\left.\frac{\partial r_{ref}}{\partial t_0}\right|_f n\delta t_0, \tag{5}$$

$$f' = f + \left.\frac{\partial f}{\partial a}\right|_f \delta a + \left.\frac{\partial f}{\partial e}\right|_f \delta e + \frac{1}{n}\left.\frac{\partial f}{\partial t_0}\right|_f n\delta t_0. \tag{6}$$

where

$$\frac{\partial r_{ref}}{\partial a} = \frac{r_{ref}}{a} - \frac{3}{2}\frac{n(t-t_0)e\sin f}{\sqrt{1-e^2}},\ \frac{\partial r_{ref}}{\partial e} = -a\cos f,\ \frac{\partial r_{ref}}{\partial t_0} = -\sqrt{\frac{\mu}{p}}e\sin f, \tag{7}$$

$$\begin{aligned} &\frac{\partial f}{\partial a} = -\frac{3}{2}\frac{n(t-t_0)a\sqrt{1-e^2}}{r_{ref}^2},\ \frac{\partial f}{\partial e} = \frac{(2+e\cos f)\sin f}{1-e^2}, \\ &\frac{\partial f}{\partial t_0} = -\sqrt{\frac{\mu}{p^3}}(1+e\cos f)^2 \end{aligned}. \tag{8}$$

Then the relative position of satellite,

$$\mathbf{r} = \mathbf{r}_{sat} - \mathbf{r}_{ref}. \tag{9}$$

can be derived as follows;

$$x = -a\delta e\cos f + \sqrt{\frac{\mu}{p}}\delta t_0 e\sin f + \left(\frac{r_{ref}}{a} - \frac{3}{2}\frac{n(t-t_0)e\sin f}{\sqrt{1-e^2}}\right)\delta a. \tag{10}$$

$$y = r_{ref}(2 + e\cos f)\sin f \frac{\delta e}{1-e^2} + r_{ref}\delta\omega + r_{ref}\sqrt{\frac{\mu}{p^3}}\delta t_0(1+e\cos f)^2$$
$$-\frac{3}{2}\frac{n(t-t_0)a\sqrt{1-e^2}}{r_{ref}}\delta a \quad . \tag{11}$$

$$z = r_{ref}\delta i \sin(f - \Omega') \,. \tag{12}$$

## 3. ONE/TWO-DIMENSIONAL GEOMETRY MAINTANANCE

In the previous section, the relative can be represented by orbital parameters. By using these results we can design the orbits that can maintain the one/two dimensional geometry.

### 3.1 Inner Product of Relative Position

With the variable conversion of

$$\alpha = \frac{\delta e}{1-e^2},\ \beta = -\sqrt{\frac{\mu}{p^3}}\delta t_0,\ \gamma = \delta\omega,\ \delta = \delta i\cos\Omega',\ \varepsilon = \delta i \sin\Omega', \tag{13}$$

the inner product of relative position can derived as following form;

$$\frac{\mathbf{r}_i^T\mathbf{r}_j}{r_{ref}^2} = p_0^{ij} + \sum_{k=1}^{3}\left[p_k^{ij}\cos kf + q_k^{ij}\sin kf\right], \tag{14}$$

where

$$p_0^{ij} = \left(\frac{e^2}{2}+\frac{5}{2}\right)\alpha_i\alpha_j + \left(\frac{e^4}{2}+\frac{7e^2}{2}+1\right)\beta_i\beta_j + \gamma_i\gamma_j$$
$$-\left(\frac{e^2}{2}+1\right)\left(\beta_i\gamma_j+\beta_j\gamma_i\right) + \frac{1}{2}\delta_i\delta_j + \frac{1}{2}\varepsilon_i\varepsilon_j$$
$$p_1^{ij} = \frac{5}{2}e\alpha_i\alpha_j + \left(\frac{7e^3}{2}+4e\right)\beta_i\beta_j + \gamma_i\gamma_j - 2e\left(\beta_i\gamma_j+\beta_j\gamma_i\right)$$
$$p_2^{ij} = \left(\frac{e^2}{2}-\frac{3}{2}\right)\alpha_i\alpha_j + \left(\frac{e^4}{2}+\frac{5e^2}{2}\right)\beta_i\beta_j - \frac{1}{2}e^2\left(\beta_i\gamma_j+\beta_j\gamma_i\right)$$
$$-\frac{1}{2}\delta_i\delta_j + \frac{1}{2}\varepsilon_i\varepsilon_j$$
$$p_3^{ij} = -\frac{e}{2}\alpha_i\alpha_j + \frac{e^3}{2}\beta_i\beta_j$$
$$q_1^{ij} = -\left(\frac{e^2}{2}+2\right)\left(\alpha_i\beta_j+\alpha_j\beta_i\right) + 2\left(\alpha_i\gamma_j+\alpha_j\gamma_i\right)$$
$$q_2^{ij} = -2e\left(\alpha_i\beta_j+\alpha_j\beta_i\right) + \frac{e}{2}\left(\alpha_i\gamma_j+\alpha_j\gamma_i\right) - \frac{1}{2}\left(\delta_i\varepsilon_j+\delta_j\varepsilon_i\right)$$
$$q_3^{ij} = -\frac{e^2}{2}\left(\alpha_i\beta_j+\alpha_j\beta_i\right) \quad . \tag{15}$$

The number of terms is seven (When the reference orbit is circle, the number of terms is five).

### 3.2 Distance Maintenance

In general, it is difficult to maintain the distance between the satellites on elliptic orbits. But by using eq. (14), we can design such orbits. When the distance between the satellites is requested to be kept at $D$, the following equation should be satisfied;

$$p_0 + \sum_{k=1}^{3}\left[ p_k \cos kf + q_k \sin kf \right] = \frac{D^2}{r_{ref}^2}. \tag{16}$$

By substituting eq. (1) to eq. (16), the right-hand term can be transformed.

$$\frac{D^2}{r_{ref}^2} = \left(1+\frac{e^2}{2}\right)\frac{D^2}{p^2} + \frac{2eD^2}{p^2}\cos f + \frac{e^2D^2}{2p^2}\cos 2f . \tag{17}$$

Then, the conditions for distance maintenance become as follows;

$$p_0 = \left(1+\frac{e^2}{2}\right)\frac{D^2}{p^2},\ p_1 = \frac{2eD^2}{p^2},\ p_2 = \frac{e^2D^2}{2p^2},\ p_3 = q_1 = q_2 = q_3 = 0 . \tag{18}$$

As the number of orbital parameter is 5 and the number of condition is 7, the solution does not exist. By neglecting the coefficients of the high frequency terms, $p_3$ and $q_3$, the approximate solution can be obtained.

$$\begin{aligned} &\alpha = 0,\ \beta = \pm\frac{2D}{p}\frac{1}{\sqrt{7e^2+8\pm 4\sqrt{3e^4-9e^2+4}}}, \\ &\gamma^2 = \left(\frac{3e^4}{4}-\frac{9e^2}{4}+1\right)\beta^2,\ \delta^2 = \left(-\frac{3e^4}{4}+3e^2\right)\beta^2,\ \varepsilon = 0 \end{aligned} \tag{19}$$

This solution exists only under the following condition;

$$e < \sqrt{\frac{1}{6}(9+\sqrt{33})} \approx 0.7366 . \tag{20}$$

Figure 2 shows the results of numerical simulation. When the eccentricity is small, the distance is almost constant. In this study, the orbit that maintains the distance between the satellites is derived analytically, but another approach is described in ref. [3] and [4].

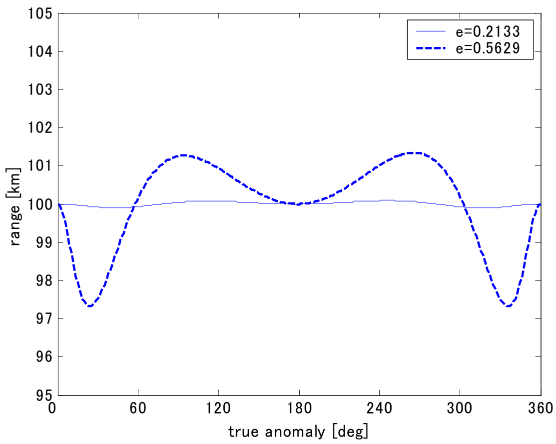


Figure 2: Simulation results of distance maintenance

### 3.3 Similar Geometry Maintenance

In order to maintain the two-dimensional similar geometry, the following conditions are considered;

$$\mathbf{r}_1 \perp \mathbf{r}_2 \Rightarrow \mathbf{r}_1^T \mathbf{r}_2 = 0, \quad |\mathbf{r}_1| = |\mathbf{r}_2|. \tag{21}$$

The former condition means the maintenance of the angle of relative positions and the latter means the equalization of the distance. The following conditions can be derived from eq. (21);

$$\begin{aligned} &p_0^{12} = p_1^{12} = \cdots = q_3^{12} = 0, \\ &p_0^{11} = p_0^{22}, \quad p_1^{11} = p_1^{22}, \quad \cdots, \quad q_3^{11} = q_3^{22} \end{aligned}. \tag{22}$$

As the number of orbital parameter is 10 and the number of condition is 14, the solution does not exist. By neglecting the high frequency term and introducing two free parameters, the solutions are obtained.

$$\begin{aligned} &\alpha_1 = -\sigma \cos\phi, \quad \beta_1 = \frac{1}{e}\sigma \sin\phi, \quad \gamma_1 = -\frac{1}{e}\left(\frac{e^2}{4}+1\right)\sigma \sin\phi, \\ &\delta_1 = \pm\sqrt{3-\frac{e^2}{4}}\sigma \sin\phi, \quad \varepsilon_1 = \mp\sqrt{3-\frac{e^2}{4}}\cos\phi \\ &\alpha_2 = -\sigma \cos\left(\phi+\frac{\pi}{2}\right), \quad \beta_2 = \frac{1}{e}\sigma \sin\left(\phi+\frac{\pi}{2}\right), \quad \gamma_2 = -\frac{1}{e}\left(\frac{e^2}{4}+1\right)\sigma \sin\left(\phi+\frac{\pi}{2}\right), \\ &\delta_2 = \pm\sqrt{3-\frac{e^2}{4}}\sigma \sin\left(\phi+\frac{\pi}{2}\right), \quad \varepsilon_2 = \mp\sqrt{3-\frac{e^2}{4}}\cos\left(\phi+\frac{\pi}{2}\right) \end{aligned}. \tag{23}$$

By using these parameters, the two-dimensional geometry can be maintained.

The result of numerical simulation is shown in Figure 3. Although the size changes, the letter ‘E’ form is maintained.

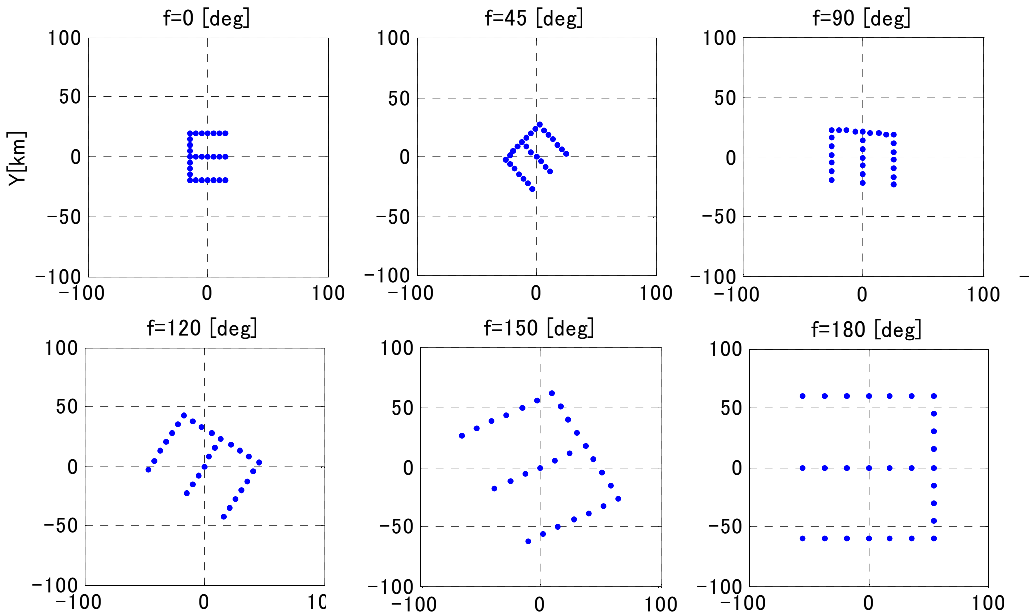


Figure 3: Maintenance of “E” formation.

## 4. THREE DIMENSIONAL GEOMETRY MAINTANANCE FOR SCOPE

In this section, the three dimensional geometry maintenance methods are represented.

### 4.1 Relation between the Spatial Resolution and Geometry

The objective of SCOPE mission is to investigate the spatial structure of plasma and magnetic field. In short, the estimation of spatial derivatives of various observations is required. At least four satellites are necessary for estimation of first order spatial derivatives. And the accuracy of estimation depends on the geometry. For example, when the four satellites are placed on the same plane, the estimation of spatial derivatives is impossible. Therefore, the geometry is very important for spatial observation.

Here we assume that the value of observation is the function of satellite's position.

$$Y = f(x, y, z)\,. \tag{24}$$

When the spatial derivative

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)^T \tag{25}$$

is uniform on the observation area, the observation value of satellite on $\mathbf{r} = (x, y, z)^T$ can be written as follows;

$$Y = Y_0 + \mathbf{r}^T \nabla f\,. \tag{26}$$

$Y_0$ is the observation value at the origin. Here we consider one mother satellite on the origin and three daughter satellites. The difference of observations between mother and *i*-th daughter satellite is

$$Z_i = Y_i - Y_0 = \mathbf{r}_i^T \nabla f\,. \tag{27}$$

Then, the $\nabla f$ can be estimated as follows;

$$\hat{\nabla} f = \left(\sum_{i=1}^{3} \mathbf{r}_i \mathbf{r}_i^T\right)^{-1} \left(\sum_{i=1}^{3} \mathbf{r}_i Z_i\right). \tag{28}$$

When the accuracy of observation depends on the distance between the satellites, $Z_i$ can be written as follows;

$$Z_i = Y_i - Y_0 = \mathbf{r}_i^T \nabla f + \|\mathbf{r}_i\| v_i\,, \tag{29}$$

where $v_i$ is white noise whose variance is $\sigma^2$. In this case, the covariance of estimation value is

$$P = \left(\sum_{i=1}^{3} \frac{\mathbf{r}_i \mathbf{r}_i^T}{\|\mathbf{r}_i\|^2}\right)^{-1} \sigma^2\,. \tag{30}$$

The geometry that minimize the $J = \sqrt{\mathrm{Tr}(P)}$ is the good formation for the spatial observation and it is shown in the figure 4 (left-hand figure). If the observation differences between daughter satellites are available, the best geometry for spatial observation is tetrahedron shown in figure 4(right-hand figure).

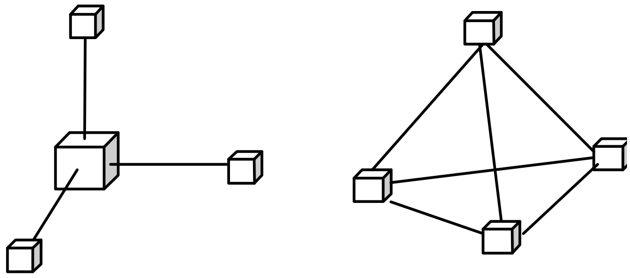

Figure 4: Suitable formation for spatial observation

**4.2 Right-Angled Baselines**

Here we consider the formation that constitutes three baselines which cross perpendicularly (Figure 4, left). We can easily constitute a baseline in the direction of $Z$ axis by a satellite whose inclination is different for a while from the mother satellite. So what we should do is to constitute two baselines which cross perpendicularly on the orbital plane of the mother satellite. The relative position of the daughter satellite on the orbital plane of the mother satellite is dependent on three parameters, $\delta e, \delta\omega$ and $\delta t_0$. Here new parameters $R, \phi$ and $k$ are introduced.

$$\frac{-\delta e}{1-e^2} = 2R\sin\phi, \quad e\sqrt{\frac{\mu}{p^3}}\delta t_0 = 2R\cos\phi, \quad -\sqrt{\frac{\mu}{p^3}}\delta t_0 - \delta\omega = kR\,. \tag{31}$$

Then the relative position can be written as follows;

$$\frac{\mathbf{r}}{r} = R\left[(3+2e\cos f)\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix} + \begin{pmatrix}\cos(2f-\pi-\phi)\\ \sin(2f-\pi-\phi)\end{pmatrix} + k\begin{pmatrix}\cos(f-\pi/2)\\ \sin(f-\pi/2)\end{pmatrix}\right]. \tag{32}$$

As this equation indicates, the parameter $R$ determines the size of relative motion and both $\phi$ and $k$ determine the form of relative motion.

As described above, it is impossible to maintain three-dimensional formation over the orbit. However, it is not necessary to maintain the formation at all the places on orbit in plasma physics as the focused area is near the apogee. Then here we design the orbit that maintains the three-dimensional geometry at $170 \le f \le 190$ [deg]. The orbital parameters of the reference orbit (mother sat.) are shown in Table 1.

Table 1: Orbital parameters of mother satellite

| | |
|---|---|
| radius of perigee $r_p$ : | $3R_e$ |
| radius of apogee $r_a$ : | $30R_e$ |
| semi-major axis $a$ : | 105237 [km] |
| eccentricity $e$ : | 0.8182 |
| inclination $i$ : | 0 [deg] |

$R_e$: radius of earth

For the design of the orbit that maintains two right-angled baselines, six parameters should be considered ($R_1, R_2, \phi_1, \phi_2, k_1$ and $k_2$). To reduce the number of parameters, the following conditions are considered;

$$R_1 = R_2 = R,\quad \phi_1 = \phi_2 = \phi,\quad k_1 = -k_2 = k\,. \tag{33}$$

These equations are given from the symmetry of two relative position vectors at the apogee. Therefore we can design the formation shape by considering only two parameters, $\phi$ and $k$ because $R$ determines only the size of formation.

By calculating $\phi$ and $k$ that minimize

$$J = \sum_{f=170^\circ}^{190^\circ} \left| \theta - \frac{\pi}{2} \right| \quad (\theta : \text{angle between two baselines})\,. \tag{34}$$

we can obtained desired orbit. $\phi$ =82.35 [deg] and $k = -2.294$ are obtained under the conditions of Table 1.

The results of numerical simulation are shown in Figure 5 and Figure 6. Figure 5 shows the histories of angle and the range of the baselines. The angle is kept at about 90 [deg]. Figure 6 shows the relative positions of the daughter satellites at $f = 170$, 180 and 190[deg] and the relative motion in inertia system. The right triangle can be maintained in inertia space.

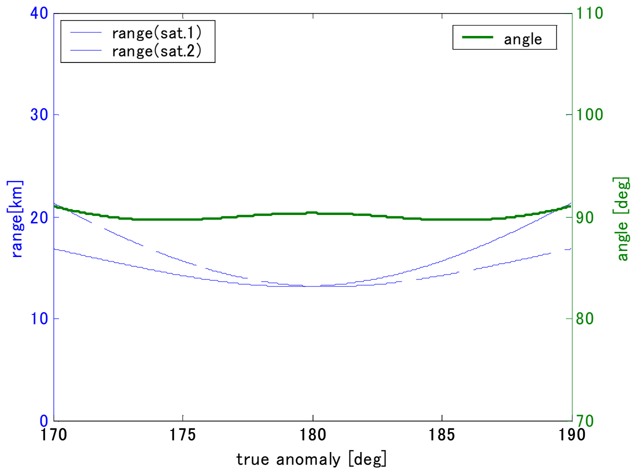


Figure 5: Range and angle of the two baselines

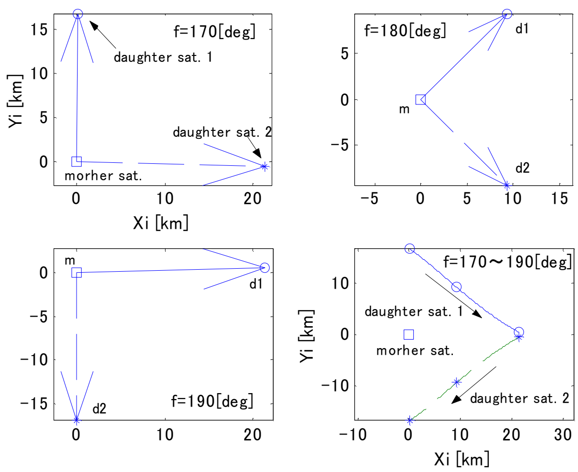


Figure 6: Maintenance of rectangular triangle

### 4.3 Tetrahedron Formation

In this section, the tetrahedron formation maintenance is indicated. As described in previous section, three satellites on the same orbital plane and one out-of-plane satellite are considered. For the maintenance of tetrahedron formation, three satellites should constitute regular triangle and out-of-plane satellite should be located above the center of the triangle. Here we assume that the orbital parameters of out-of-plane satellite are

$$a' = a, \quad e' = e + \delta e, \quad i' = \delta i, \quad \Omega' = \pm 90[\text{deg}], \quad \omega' = \mp 90[\text{deg}], \quad \delta t_0' = 0\,. \quad (35)$$

By considering eq. (33) and (35), the shape of formation can be determined by the four parameters, $\phi, k, \delta e$ and $\delta i$. As described in section 4.1, the index of special resolution is

$$P = \sqrt{\mathrm{Tr}\left[\left(\sum_{i=1}^{3} \frac{\mathbf{r}_i \mathbf{r}_i^T}{\left\|\mathbf{r}_i\right\|^2}\right)^{-1}\right]}\,. \quad (36)$$

So, by calculating the parameters that minimize

$$J = \sum_{f=170^\circ}^{190^\circ} P\,, \quad (37)$$

orbit that maintained the tetrahedron formation is obtained.

Figure 7 shows the result of numerical simulation. Although there is some distortion, the regular tetrahedron geometry is kept near the apogee.

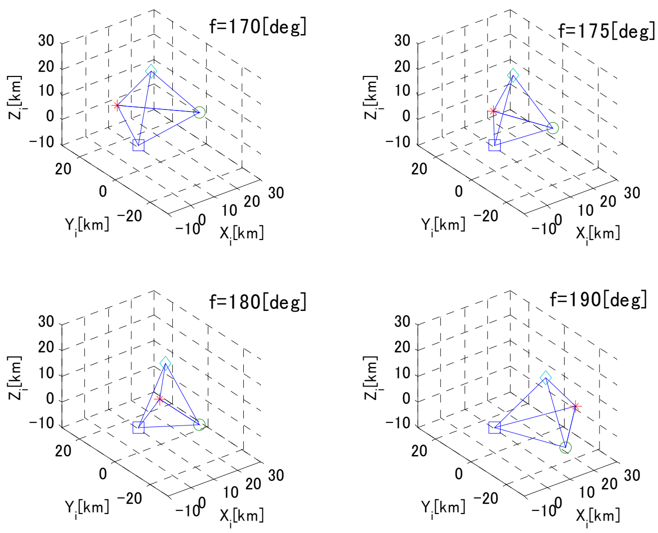


Figure 7: Maintenance of tetrahedron formation

## 5. SUMMARY

This paper shows the method of the orbital design for SCOPE mission. SCOPE mission requires the highly elliptic orbit for the observation. As is well known, it is impossible to maintain the three dimensional formation over the orbit, but the frozen property that maintains high spatial resolution near the apogee is found feasible for elliptic orbit. The orbits designed in this paper are very effective for many science missions as well as SCOPE mission.

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