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# Elements of mathematical phenomenology:
# I. Mathematical and qualitative analogies

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**Abstract**

Paper starts with short description of *Element of Mathematical Phenomenology* and *Phenomenological Mappings* published in Petrović's theory. The biographical data of Mihailo Petrović (1868-1943) is presented. Petrović was a important Serbian mathematician, one of three Poincare's doctoral students. Some of basic elements of mathematical phenomenology in mechanics are elements of non-linear-functional transformations of coordinates from one to other functional curvilinear coordinate system. Some of these elements, as it is basic vectors of tangent space of kinetic point vector position and their changes (velocity of their magnitude extensions and component angular velocities of rotations), are presented in different functional coordinate systems. Mihailo Petrović's theory contains two types of analogies: mathematical and qualitative, and in this paper third type - structural analogy is described. Taking into account large possibility for applications of all three types of analogies, numerous original examples are presented using, between other, fractional system dynamics with one degree of freedom, finite number of degrees of freedom as well as multi-body discrete continuum hybrid fractional order system dynamics. Mathematical analogies between vector models in local area of stress state, strain stare of the point in stressed and deformed deformable body as well as with vector model of the mass inertia moment state at point of rigid body, used mass inertia moment vectors coupled for pole and axis, are presented, also.

A number of theorems of energy fractional order dissipation presented in corresponding Tables, also. It is shown that applications of qualitative, structural and mathematical analogies in analysis of fractional order modes appear in analogous mechanical, electrical and biological fractional order chains, and that is very power, suitable and useful tools to reduce research models to corresponding minimal numbers, and, in same time, develop power of analysis use mathematical and qualitative analogies.

**Keywords:** Mihailo Petrović (1868-1943), mathematical phenomenology, fractional order system, generalized functions of fractional order dissipation of system energy, fractional order modes, theorem mass moment vectors, vector model, stress; strain.

## I. INTRODUCTION

Let us start with short description of *Element of Mathematical Phenomenology* [40] and *Phenomenological Mappings* [41] published in Petrović's theory and a few biographical data of Mihailo Petrović. Mihailo Petrović (1868-1943) was a important Serbian mathematician, one of three Poincaré's doctoral students. His professors were world important scientists, such as Poincaré (Jules Henri Poincaré (1854 –1912)), Appall (Paul Appell (1855 –1930)), Hermite (Charles Hermite (1822 –1901)), Picard (Charles Émile Picard (1856 –1941)), Painlevé (Paul Painlevé (1863 –1933)), Bousinesq (Joseph Valentin Boussinesq (1842 –1929)) and others. He published the books "Elements of Mathematical Phenomenology” [40] in 1911 and “Phenomenological Mapping" [41] in 1933. Evaluating Petrović’s Elements of Mathematical Phenomenology, Milutin Milanković (1879 - 1958) (author of world known and numerous cited “*Canon of Sun insulation*”) said that this book is very important, but, in time of publishing, understandable only to two scientists in Serbia, and in my opinion, also, two in America–Mihailo Pupin (1858 -1935) and Nikola Tesla (1856 – 1943), because the book was published in Serbian Language. A short presentation of this book was in French [42]. In his book [40], Petrović studied some elements of *multi-dimensional geometry*, *coupling between mechanisms and manifestation of phenomena*, *transformation of equations for phenomenon in holonomic systems*, *potential phenomena*, presented and analyzed *systems of Appel’s and Lagrange’s equations*, *actions of discontinuous causes*, *the events of occurrence as a result of its composition mechanisms*, *quantitative and qualitative images of appearances* (phenomenon), *the composition and patterns of phenomenological mechanisms*, *phenomenological analogies*, *mathematical analogies and qualitative analogy.*

A short presentation of this book was in French, titled by *Mecanismes communs aux phenomens disparates*, Paris 1921 [42].

Petrović presented properties of the theory of the viva forces and their phenomenological consequences. He pointed out that there are many differences between pure geometrical and dynamical properties of a system. As pioneer in Mathematical Phenomenology and Phenomenological Mapping, Petrović and Petrović's scientific work was inspiration for this paper. Nowadays, numerous mathematicians working in different areas of sciences are using mathematical formalism. Mathematical formalism in technical sciences is very dangerous without knowledge of forces and energy changes and their interactions.

Using elements of mathematical phenomenology and in particular different types of analogies, qualitative, structural and mathematical elements, it is possible to make precise or approximate phenomenological mappings of phenomena [40] from global to local area of system kinetic parameters, but is necessary to add corresponding conditions of restrictions. Also, it is possible to make analogy between two different phenomena in two or more systems from disparate areas of science and identify equal or similar properties expressed by elements of mathematical phenomenology.

It is possible to transfer knowledge from one area of science to another, integrate research and create a basic kernel of science for education of next generations of civilization using minimum energy and time of study.

## II. ABSTRACTION OF REAL SYSTEM TO THE PHYSICAL, CHEMICAL OR BIOLOGICAL AND MATHEMATICAL MODEL

Abstraction from real system dynamics to the corresponding model usually, with some suppositions and valuable approximations as well as with numerous conditions of restrictions in applications, gives different models with non-linear and corresponding simple linear dynamics good satisfying description of real system behaviors. A present day, linear models, as well as linearized models of different real dynamics abstractions are acceptable for use in numerous applications. Corresponding mathematical descriptions of linear system dynamics are very good and applicable for investigations and analysis of a local dynamics of real system in a constrained area of kinetic parameters. But, when questions are about some systems with complex phenomena linear descriptions are not satisfying and then appear numerous problems. First main problem is that analytical methods for solving problems of mathematical descriptions by non-linear algebra equations, nonlinear ordinary differential equations, nonlinear partial differential equations or fractional order differential equations, integro-differential equations and others, in analytical form are not founded.

Power computers and fast developments of numerous numerical methods opened a large field for obtaining numerous data of possible non-linear phenomena, on the level of numerous numerical experiments with numerical data, but results are approximate particular numerical solutions or multi-parametric series of particular approximate solutions. On the basis of the numerous approximate particular solutions obtained by numerical experiments are possible to obtain some qualitative conclusions. But, at same time, questions about errors in numerical approximations and precisions of qualitative conclusions appear.

In present day, it is necessary that scientists pay attentions to analytical methods for investigation non-linear phenomena and for obtaining analytical tools for conclusions of qualitative properties of non-linear phenomena and compare obtained results in results obtained in different area of sciences and to identified analogies between these results,qualitative, structural and mathematical and possibilities of phenomenological mappings between same types of non-linear phenomena in dynamics of physically or bio-dynamical disparate systems (see References [2,3] and [30]).

## III. ELEMENTS OF MATHEMATICAL PHENOMENOLOGY IN NON-LINEAR TRANSFORMATION OF COORDINATES

### III.1. ELEMENTS OF MATHEMATICAL PHENOMENOLOGY IN NON-LINEAR TRANSFORMATION OF COORDINATES.

Let's consider two systems of variables $q^i$ and $\bar{q}^i$ with known non-linear relation between first and second in the following forms: $\bar{q}^i = \bar{q}^i(q)$ or $q^i = q^i(\bar{q})$. Last two relations are inverse one to other. Last relations satisfy condition, that functional determinate (Jacobean) for each of these is different from zero, $\bar{\mathbf{J}} = \left|\frac{\partial \bar{q}^i(q)}{\partial q^k}\right| \neq 0$ and $\mathbf{J} = \left|\frac{\partial q^i(\bar{q})}{\partial \bar{q}^k}\right| \neq 0$.

For the case of linear, affine transformation, functional determinant (Jacobean) is determinant of transformation matrix in the form $\left|\frac{\partial x^j}{\partial y^i}\right| = |\mathbf{A}| = \left|a_i^j\right| = const$, but is constant. That is a special case of general functional (non-linear, generalized) transformation. For the case of the general functional (non-linear, generalized) transformation, it is possible to write the following relations (see References [45-48]):

$$d\bar{q}^i = \frac{\partial \bar{q}^i(q)}{\partial q^k} dq^k \quad \text{or} \quad dq^i = \frac{\partial q^i(\bar{q})}{\partial \bar{q}^k} d\bar{q}^k \tag{1}$$

For the case of general, functional (non-linear, generalized) transformation, straight lines mapped into curvilinear lines and only differential elements of curvilinear lines are mapped by affine mapping [46,48,17]. It is then possible to make a conclusion: in results of nonlinear mapping of straight lines curvilinear lines appear. For detail about *difference between affine (linear) and nonlinear transformation of coordinates from one to others* see References [46], [48] and [17].

### III.2. NON-LINEAR TRANSFORMATIONS OF COORDINATES AND BASIC VECTORS IN GENERALIZED FUNCTIONAL TANGENT SPACES OF POSITION VECTOR OF A KINEMATIC POINT

Let's consider, in curvilinear coordinate system, with coordinates denoted by $q^i$, for difference of linear, affine coordinates denoted by $x^i$, corresponding vectors

$$\vec{g}_i(q) = \frac{\partial \vec{r}}{\partial q^i}, \quad i = 1,2,3,\ldots,N \qquad (2)$$

which are defined as *basic vectors* of curvilinear generalized functional coordinate system of the *tangent space of kinetic point position vector* $\vec{r}(q)$, with source pole in end of position vector at moving kinetic point $\mathbf{N}(q^1, q^2, q^3)$. By using these basic vectors $\vec{g}_i(q) = \frac{\partial \vec{r}}{\partial q^i}$ it is possible to define corresponding coordinate system with curvilinear coordinates $q^i$, but with source pole in source pole of position vector $\vec{r}(q)$. Then it is visible that curvilinear coordinates (parameters) $q^i$ are not coordinates of position vector $\vec{r}(q)$, and as coordinates of position vector $\vec{r}(q)$ in **c**urvilinear coordinate system are introduced in the form $r^i(q)$ as its contra-variant coordinates. It is clear that these contra-variant coordinates $r^i(q)$ of position vector are non-linear functions of curvilinear coordinates (parameters) $q^i$, and also that basic vectors of corresponding curvilinear coordinate system are non-linear functions of curvilinear coordinates (parameters) $q^i$ (See Figure 1.a* and 1.b*, and for details see *References* [36], [46], [48], [17] and [19]).

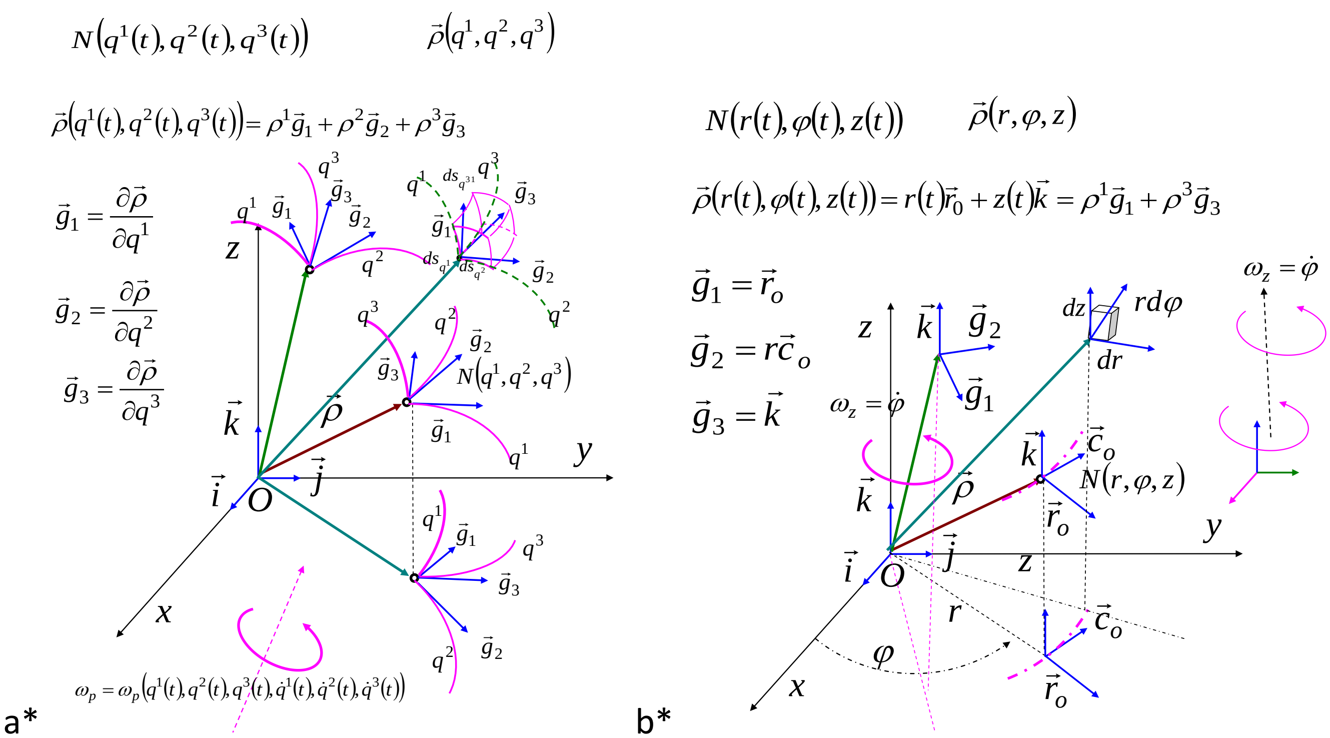



**Figure 1.** Presentation of the position vector of a kinetic point in different positions in three dimensional space, with corresponding basic vectors $\vec{g}_{(\alpha)i}$ of position vector $\vec{\rho}_{(\alpha)}(q)$ tangent space (without index $(\alpha)$ denotation of the order of point); **a*** in arbitrary generalized curvilinear orthogonal coordinates; **b***in polar cylindrical coordinate system with orthogonal curvilinear coordinates.

### III.3. ANGULAR VELOCITY AND INTENSITY CHANGE OF BASIC VECTORS OF POSITION VECTOR TANGENT SPACE OF A MATERIAL SYSTEM KINETIC POINT

In real three dimensional coordinate system, position vectors of the material–kinetic points of a material system constrained by geometrical holonomic stationary and nonstationary real constraints (*see References* [36], [46], [48], [17] and [19]), are denoted by $\vec{\rho}_{(\alpha)}(q)$, $\alpha = 1,2,3,\ldots,N$ and each as functions of generalized curvilinear coordinates $q^i_{(\alpha)}$, $\alpha = 1,2,3,\ldots,N$, $i = 1,2,3$, where $N$ is the total number of material system mass particles. Basic vectors of each kinetic point position vector tangent space are denoted by $\vec{g}_{(\alpha)i}$, $\alpha = 1,2,3,\ldots,N$, $i = 1,2,3$: (see Figure 1), and can be expressed in the following form::

$$\vec{g}_{(\alpha)i} = \frac{\partial \vec{\rho}_{(\alpha)}}{\partial q^{\,i}_{(\alpha)}}, \ \alpha = 1,2,3,\ldots,N, \quad i = 1,2,3 \qquad (3)$$

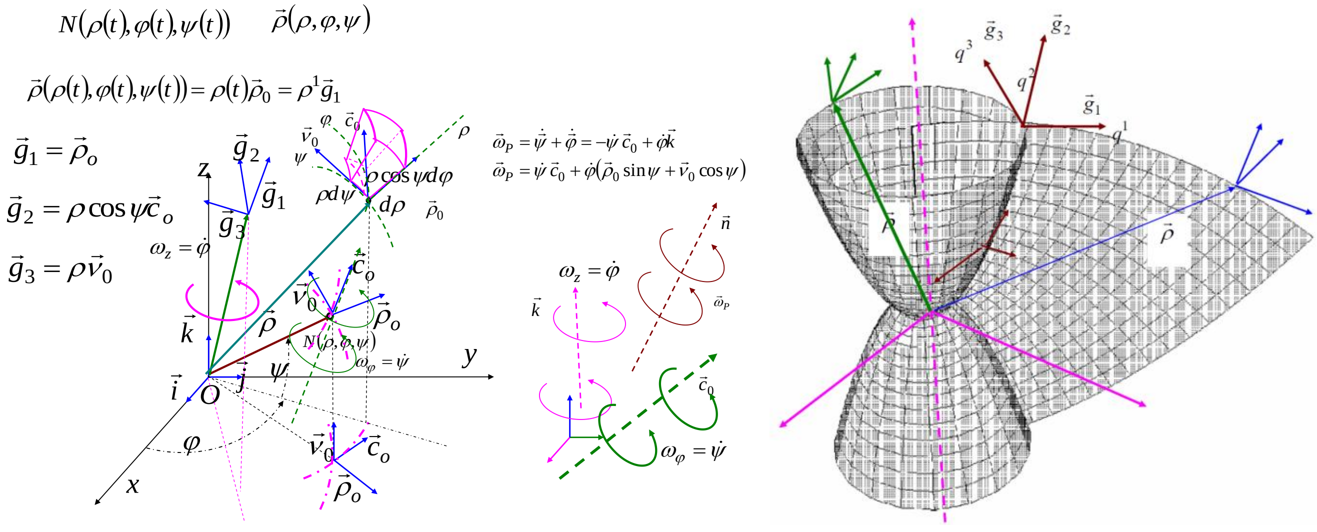
All basic vectors (3) are functions of the time depending curvilinear coordinates (parameters) $q_{(\alpha)}^{\,i}(t)$, which are nonlinear function of time $t$.

### III.4. CHANGE OF BASIC VECTORS OF POSITION VECTOR TANGENT SPACE OF KINETIC POINT IN THREE DIMENSIONAL SPACE EACH IN CURVILINEAR COORDINATE SYSTEM

Without losing generality, let's consider and determine expressions for change of basic vectors $\vec{g}_{(\alpha)i}$, $\alpha = 1,2,3,\ldots,N$, $i = 1,2,3$ of position vector $\vec{\rho}_{(\alpha)}$, $\alpha = 1,2,3,\ldots,N$ in three dimensional tangent space in curvilinear coordinate system for one kinetic point of material system.

Let's suppose that each position vector tangent space of each kinetic point $\mathbf{N}_{(\alpha)}\left(q^1_{(\alpha)}, q^2_{(\alpha)}, q^3_{(\alpha)}\right)$, is three dimensional and defined in orthogonal curvilinear coordinates $q^i_{(\alpha)}$, $\alpha = 1,2,3,\ldots,N$, $i = 1,2,3$. Then, let's to separate in expressions of the corresponding derivatives $\frac{d\vec{g}_{(\alpha)i}}{dt}$, $\alpha = 1,2,3,\ldots,N$, $i = 1,2,3$ of the basic vectors $\vec{g}_{(\alpha)i}$, $\alpha = 1,2,3,\ldots,N$, $i = 1,2,3$ of position vector

$\vec{\rho}_{(\alpha)}$, $\alpha = 1,2,3,\ldots,N$ tangent space terms which correspond to terms of the relative derivatives $\overset{*}{\vec{g}}_{(\alpha)1}$ , $\overset{*}{\vec{g}}_{(\alpha)2}$ and $\overset{*}{\vec{g}}_{(\alpha)3}$ , $\alpha = 1,2,3,\ldots,N$ (*see References* [46], [48], [39] [17] and [19].).



a*                                                                 b*

**Figure 2.** Presentation of the position vector of a kinetic point in different positions in three dimensional space, with corresponding basic vectors $\vec{g}_{(\alpha)i}$ of position vector $\vec{\rho}_{(\alpha)}(q)$ tangent space (without index $(\alpha)$ denotation of the order of point); **a*** in spherical coordinate system with orthogonal curvilinear coordinates; b* in three dimensional three parabolic coordinate system with orthogonal curvilinear coordinates.

These vector terms, $\overset{*}{\vec{g}}_{(\alpha)1}$ , $\overset{*}{\vec{g}}_{(\alpha)2}$ and $\overset{*}{\vec{g}}_{(\alpha)3}$ represent vectors of relative velocity of basic vectors extensions and it is possible to express in scalar forms as relative velocity of magnitude dilatation of each of three basic vectors of each position vector tangent space of each kinetic point in following forms (for detail see References [25,26] and [16,17]):.

$$\varepsilon_{(\alpha)1} = \frac{d\left|\vec{g}_{(\alpha)1}\right|}{\left|\vec{g}_{(\alpha)1}\right| dt} = \left(\Gamma^{1}_{(\alpha)11}\dot{q}^{1}_{(\alpha)} + \Gamma^{1}_{(\alpha)12}\dot{q}^{2}_{(\alpha)} + \Gamma^{1}_{(\alpha)13}\dot{q}^{3}_{(\alpha)}\right)$$

$$\varepsilon_{(\alpha)2} = \frac{d\left|\vec{g}_{(\alpha)2}\right|}{\left|\vec{g}_{(\alpha)2}\right| dt} = \left(\Gamma^{2}_{(\alpha)21}\dot{q}^{1}_{(\alpha)} + \Gamma^{2}_{(\alpha)22}\dot{q}^{2}_{(\alpha)} + \Gamma^{2}_{(\alpha)23}\dot{q}^{3}_{(\alpha)}\right) \qquad (4)$$

$$\varepsilon_{(\alpha)3} = \frac{d\left|\vec{g}_{(\alpha)31}\right|}{\left|\vec{g}_{(\alpha)3}\right| dt} = \left(\Gamma^{3}_{(\alpha)31}\dot{q}^{1}_{(\alpha)} + \Gamma^{3}_{(\alpha)32}\dot{q}^{2}_{(\alpha)} + \Gamma^{3}_{(\alpha)33}\dot{q}^{3}_{(\alpha)}\right),\ \alpha = 1,2,3,\ldots,N\ .$$

Other terms in each of the expressions of the corresponding derivatives $\frac{d\vec{g}_{(\alpha)i}}{dt}$, $\alpha = 1,2,3,\ldots,N$, $i = 1,2,3$ of the basic vectors $\vec{g}_{(\alpha)i}$, $\alpha = 1,2,3,\ldots,N$, $i = 1,2,3$ of position vector $\vec{\rho}_{(\alpha)}$, $\alpha = 1,2,3,\ldots,N$ tangent space are terms which represent the vector expressions of vector product between angular velocity $\vec{\omega}_{p(\alpha)i}$, $\alpha = 1,2,3,\ldots,N$, $i = 1,2,3$ of corresponding basic vector rotation and same basic vector $\vec{g}_{(\alpha)i}$, $\alpha = 1,2,3,\ldots,N$, $i = 1,2,3$. From these terms it is easier to express angular velocities of the basic vectors of position vector tangent space during motion of the corresponding kinetic point. These expressions are in the following forms:

$$\left[\vec{\omega}_{p(\alpha)1}, \vec{g}_{(\alpha)1}\right] = \vec{g}_{(\alpha)2}\left(\Gamma^{2}_{(\alpha)11}\dot{q}^{1}_{(\alpha)} + \Gamma^{2}_{(\alpha)12}\dot{q}^{2}_{(\alpha)} + \Gamma^{2}_{(\alpha)13}\dot{q}^{3}_{(\alpha)}\right) + \vec{g}_{(\alpha)3}\left(\Gamma^{3}_{(\alpha)11}\dot{q}^{1}_{(\alpha)} + \Gamma^{3}_{(\alpha)12}\dot{q}^{2}_{(\alpha)} + \Gamma^{3}_{(\alpha)13}\dot{q}^{3}_{(\alpha)}\right)$$

$$\left[\vec{\omega}_{p(\alpha)2}, \vec{g}_{(\alpha)2}\right] = \vec{g}_{(\alpha)1}\left(\Gamma^{1}_{(\alpha)21}\dot{q}^{1}_{(\alpha)} + \Gamma^{1}_{(\alpha)22}\dot{q}^{2}_{(\alpha)} + \Gamma^{1}_{(\alpha)23}\dot{q}^{3}_{(\alpha)}\right) + \vec{g}_{(\alpha)3}\left(\Gamma^{3}_{(\alpha)21}\dot{q}^{1}_{(\alpha)} + \Gamma^{3}_{(\alpha)22}\dot{q}^{2}_{(\alpha)} + \Gamma^{3}_{(\alpha)23}\dot{q}^{3}_{(\alpha)}\right) \qquad (5)$$

$$\left[\vec{\omega}_{p(\alpha)3}, \vec{g}_{(\alpha)3}\right] = \vec{g}_{(\alpha)1}\left(\Gamma^{1}_{(\alpha)31}\dot{q}^{1}_{(\alpha)} + \Gamma^{1}_{(\alpha)32}\dot{q}^{2}_{(\alpha)} + \Gamma^{1}_{(\alpha)33}\dot{q}^{3}_{(\alpha)}\right) + \vec{g}_{(\alpha)2}\left(\Gamma^{2}_{(\alpha)31}\dot{q}^{1}_{(\alpha)} + \Gamma^{2}_{(\alpha)32}\dot{q}^{2}_{(\alpha)} + \Gamma^{3}_{(\alpha)33}\dot{q}^{3}_{(\alpha)}\right),\quad \alpha = 1,2,3,\ldots,N$$

In previous, presented expressions, denotations $\vec{\omega}_{p(\alpha)1}$, $\vec{\omega}_{p(\alpha)2}$ and $\vec{\omega}_{p(\alpha)3}$ present angular velocities of the basic vectors of a position vector tangent space during kinetic material point motion. In previous expressions (5), $\Gamma^{k}_{(\alpha)ij}$, $\alpha = 1,2,3,\ldots,N$, $i,j,k = 1,2,3$ are Christoffel's symbols of the second kind, and $\Gamma_{(\alpha)ij,k}$, $\alpha = 1,2,3,\ldots,N$, $i,j,k = 1,2,3$ Christoffel's symbols of the first kind in corresponding curvilinear coordinate system of vector position tangent space of corresponding material kinetic point.

**Table 1.** Examples of the change of basic vectors of the position vector tangent space during kinetic point motion expressed in different curvilinear coordinate systems

| | The polar-cylindrical curvilinear coordinate systems of kinetic points $N_{(\alpha)}\left(r_{(\alpha)},\varphi_{(\alpha)},z_{(\alpha)}\right)$ $\alpha=1,2,3,...,N$ tangent spaces | The spherical curvilinear coordinate systems of kinetic points $N_{(\alpha)}\left(\rho_{(\alpha)},\varphi_{(\alpha)},\vartheta_{(\alpha)}\right)$, $\alpha=1,2,3,...,N$ tangent spaces | The three dimensional three parabolic curvilinear coordinate systems of kinetic points $N_{(\alpha)}\left(\xi_{(\alpha)},\eta_{(\alpha)},\varphi_{(\alpha)}\right)$ $\alpha=1,2,3,...,N$, tangent spaces |
|---|---|---|---|
| $N_{(\alpha)}\left(r_{(\alpha)},\varphi_{(\alpha)},z_{(\alpha)}\right)$<br>$q^{(\alpha)i}$, $i=1,2,3$<br>$\alpha=1,2,3,...,N$ | $r_{(\alpha)},\varphi_{(\alpha)},z_{(\alpha)}$, $\alpha=1,2,3,...,N$ | $\rho_{(\alpha)},\varphi_{(\alpha)},\vartheta_{(\alpha)}$, $\alpha=1,2,3,...,N$ | $\xi_{(\alpha)},\eta_{(\alpha)},\varphi_{(\alpha)}$, $\alpha=1,2,3,...,N$ |
| Vector position of kinetic point<br>$\vec{\rho}_{(\alpha)}\left(q^{(\alpha)1},q^{(\alpha)2},q^{(\alpha)3}\right)$<br>$\alpha=1,2,3,...,N$ | $\vec{\rho}_{(\alpha)}\left(r_{(\alpha)},\varphi_{(\alpha)},z_{(\alpha)}\right)=r_{(\alpha)}(t)\vec{r}_{(\alpha)0}\left(\varphi_{(\alpha)},z_{(\alpha)}\right)+z_{(\alpha)}(t)\vec{k}_{(\alpha)}$,<br>$\alpha=1,2,3,...,N$ | $\vec{\rho}_{(\alpha)}\left(\rho_{(\alpha)},\varphi_{(\alpha)},\vartheta_{(\alpha)}\right)=\rho_{(\alpha)}\vec{\rho}_{(\alpha)0}\left(\varphi_{(\alpha)},\vartheta_{(\alpha)}\right)$<br>$\alpha=1,2,3,...,N$ | $\vec{\rho}_{(\alpha)}\left(\xi_{(\alpha)},\eta_{(\alpha)},\varphi_{(\alpha)}\right)=\rho_{(\alpha)}{}^{i}\left(\xi_{(\alpha)},\eta_{(\alpha)},\varphi_{(\alpha)}\right)\vec{g}_{(\alpha)i}\left(\xi_{(\alpha)},\eta_{(\alpha)},\varphi_{(\alpha)}\right)$ |
| Basic vectors of position vector tangent space<br>$\vec{g}_{(\alpha)i}\left(q^{(\alpha)1},q^{(\alpha)2},q^{(\alpha)3}\right)$<br>$\alpha=1,2,3,...,N$ | $\vec{g}_{(\alpha)1}=\vec{r}_{(\alpha)o}$<br>$\vec{g}_{(\alpha)1}=\vec{i}\cos\varphi_{(\alpha)}+\vec{j}\sin\varphi_{(\alpha)}$<br>$\vec{g}_{(\alpha)2}=r_{(\alpha)}\vec{c}_{(\alpha)o}$<br>$\vec{g}_{(\alpha)2}=r_{(\alpha)}\left(-\vec{i}\sin\varphi_{(\alpha)}+\vec{j}\cos\varphi_{(\alpha)}\right)$<br>$\vec{g}_{(\alpha)3}=\vec{k}$ | $\vec{g}_{(\alpha)1}=\vec{\rho}_{(\alpha)0}$<br>$\vec{g}_{(\alpha)1}=\vec{i}\cos\vartheta_{(\alpha)}\cos\varphi_{(\alpha)}+\vec{j}\cos\vartheta_{(\alpha)}\sin\varphi_{(\alpha)}+\vec{k}\sin\vartheta_{(\alpha)}$<br>$\vec{g}_{(\alpha)2}=\rho_{(\alpha)}\cos\psi_{(\alpha)}\vec{c}_{(\alpha)o}$<br>$\vec{g}_{(\alpha)2}=\rho_{(\alpha)}\cos\psi_{(\alpha)}\left(-\vec{i}\sin\varphi_{(\alpha)}+\vec{j}\cos\varphi_{(\alpha)}\right)$<br>$\vec{g}_{(\alpha)3}=\rho_{(\alpha)}\vec{v}_{(\alpha)0}$<br>$\vec{g}_{(\alpha)3}=\rho_{(\alpha)}\left(-\vec{i}\sin\vartheta_{(\alpha)}\cos\varphi_{(\alpha)}-\vec{j}\sin\vartheta_{(\alpha)}\sin\varphi_{(\alpha)}+\vec{k}\cos\vartheta_{(\alpha)}\right.$ | $\vec{g}_{(\alpha)1}=\vec{g}_{(\alpha)\xi}=\frac{\partial\vec{\rho}_{(\alpha)}}{\partial\xi_{(\alpha)}}=\eta_{(\alpha)}\cos\varphi_{(\alpha)}\vec{i}+\eta_{(\alpha)}\sin\varphi_{(\alpha)}\vec{j}+\xi_{(\alpha)}\vec{k}$<br>$\vec{g}_{(\alpha)2}=\vec{g}_{(\alpha)\eta}=\frac{\partial\vec{\rho}_{(\alpha)}}{\partial\eta_{(\alpha)}}=\xi_{(\alpha)}\cos\varphi_{(\alpha)}\vec{i}+\xi_{(\alpha)}\sin\varphi_{(\alpha)}\vec{j}-\eta_{(\alpha)}\vec{k}$<br>$\vec{g}_{(\alpha)3}=\vec{g}_{(\alpha)\varphi}=\frac{\partial\vec{\rho}_{(\alpha)}}{\partial\varphi_{(\alpha)}}=-\xi_{(\alpha)}\eta_{(\alpha)}\sin\varphi_{(\alpha)}\vec{i}+\xi_{(\alpha)}\eta_{(\alpha)}coas\varphi_{(\alpha)}\vec{j}$ |
| Contravaruant coordinates of vector position<br>$\rho_{(\alpha)}{}^{i}\left(q^{(\alpha)1},q^{(\alpha)2},q^{(\alpha)3}\right)$<br>$\alpha=1,2,3,...,N$,<br>$i=1,2,3$ | $\rho_{(\alpha)1}=\rho_{(\alpha)r}=r_{(\alpha)}(t)$<br>$\rho_{(\alpha)2}=\rho_{(\alpha)\varphi}=0$<br>$\rho_{(\alpha)3}=\rho_{(\alpha)z}=z_{(\alpha)}(t)_{(\alpha)}$ | $\rho_{(\alpha)1}=\rho_{(\alpha)\rho}=\rho_{(\alpha)}(t)$<br>$\rho_{(\alpha)2}=\rho_{(\alpha)\varphi}=0$<br>$\rho_{(\alpha)3}=\rho_{(\alpha)\vartheta}=0$ | $\rho_{(\alpha)}{}^{i}=\frac{1}{\left\vert\vec{g}_{(\alpha)i}\right\vert}\left(\vec{\rho}_{(\alpha)},\frac{\vec{g}_{(\alpha)i}}{\left\vert\vec{g}_{(\alpha)i}\right\vert}\right)$, $\alpha=1,2,3,...,N$, $i=1,2,3$ |
| Dilatation of the basic vector of position vector tangent space<br>$\overset{*}{\vec{g}}_{(\alpha)i}$ $\alpha=1,2,3,...,N$,<br>$i=1,2,3$<br>$\varepsilon_{(\alpha)i}$ | $\overset{*}{\vec{g}}_{(\alpha)r}=0$, $\varepsilon_{(\alpha)r}=0$<br>$\overset{*}{\vec{g}}_{(\alpha)\varphi}=\dot{r}_{(\alpha)}\vec{c}_{(\alpha)0}=\frac{\dot{r}_{(\alpha)}}{r_{(\alpha)}}\vec{g}_{(\alpha)\varphi}$<br>$\varepsilon_{(\alpha)\varphi}=\frac{d\left\vert\vec{g}_{(\alpha)\varphi}\right\vert}{\left\vert\vec{g}_{(\alpha)\varphi}\right\vert dt}=\frac{dr_{(\alpha)}}{r_{(\alpha)}dt}$<br>$\overset{*}{\vec{g}}_{(\alpha)z}=0$, $\varepsilon_{(\alpha)z}=0$ | $\overset{*}{\vec{\rho}}_{(\alpha)\rho}=0$, $\varepsilon_{(\alpha)\rho}=0$<br>$\overset{*}{\vec{g}}_{(\alpha)\varphi}=\vec{c}_{(\alpha)0}\left(\dot{\rho}_{(\alpha)}\cos\vartheta_{(\alpha)}-\rho_{(\alpha)}\dot{\vartheta}_{(\alpha)}\sin\vartheta_{(\alpha)}\right)$<br>$\overset{*}{\vec{g}}_{(\alpha)\varphi}=\frac{1}{\rho_{(\alpha)}\cos\vartheta_{(\alpha)}}\left(\dot{\rho}_{(\alpha)}\cos\vartheta_{(\alpha)}-\rho_{(\alpha)}\dot{\vartheta}_{(\alpha)}\sin\vartheta_{(\alpha)}\right)\vec{g}_{(\alpha)\varphi}$<br>$\overset{*}{\vec{g}}_{(\alpha)\vartheta}=\dot{\rho}_{(\alpha)}\vec{v}_{(\alpha)0}=\frac{\dot{\rho}_{(\alpha)}}{\rho_{(\alpha)}}\vec{g}_{(\alpha)\vartheta}$ | $\overset{*}{\vec{g}}_{(\alpha)\xi}=\frac{1}{\left(\eta_{(\alpha)}^{2}+\xi_{(\alpha)}^{2}\right)}\left\langle\dot{\eta}_{(\alpha)}\eta_{(\alpha)}+\dot{\xi}_{(\alpha)}\xi_{(\alpha)}\right\rangle\vec{g}_{(\alpha)\xi}$<br>$\overset{*}{\vec{g}}_{(\alpha)\eta}=\frac{1}{\left(\eta_{(\alpha)}^{2}+\xi_{(\alpha)}^{2}\right)}\left\langle\dot{\eta}_{(\alpha)}\eta_{(\alpha)}+\dot{\xi}_{(\alpha)}\xi_{(\alpha)}\right\rangle\vec{g}_{(\alpha)\eta}$<br>$\overset{*}{\vec{g}}_{(\alpha)\varphi}=\frac{1}{\xi_{(\alpha)}^{2}\eta_{(\alpha)}^{2}}\left\langle\dot{\xi}_{(\alpha)}\eta_{(\alpha)}^{2}\xi_{(\alpha)}+\xi_{(\alpha)}^{2}\dot{\eta}_{(\alpha)}\eta_{(\alpha)}\right\rangle\vec{g}_{(\alpha)\varphi}$ |
| Component angular velocity of of the basic vector of position vector tangent space<br>$\vec{\omega}_{(\alpha)i}$, $\alpha=1,2,3,...,N$<br>$i=1,2,3$ | $\vec{\omega}_{P(\alpha)r}=\dot{\varphi}_{(\alpha)}\vec{k}$<br>$\vec{\omega}_{P(\alpha)\varphi}=\dot{\varphi}_{(\alpha)}\vec{k}$<br>$\vec{\omega}_{P(\alpha)z}=0$ or $\vec{\omega}_{P(\alpha)z}=\dot{\varphi}_{(\alpha)}\vec{k}$ | $\vec{\omega}_{P(\alpha)\rho}=\dot{\vartheta}_{(\alpha)}\vec{c}_{(\alpha)0}+\dot{\varphi}_{(\alpha)}\vec{k}$<br>$\vec{\omega}_{P(\alpha)\rho}=-\frac{\dot{\vartheta}_{(\alpha)}}{\rho_{(\alpha)}\cos\vartheta_{(\alpha)}}\vec{g}_{(\alpha)\varphi}+\dot{\varphi}_{(\alpha)}\left(\vec{g}_{(\alpha)\rho}\sin\vartheta_{(\alpha)}+\frac{1}{\rho}\vec{g}_{(\alpha)\vartheta}\cos\vartheta_{(\alpha)}\right)$<br>$\vec{\omega}_{P(\alpha)\rho}=\vec{\omega}_{P(\alpha)\varphi}=\vec{\omega}_{P(\alpha)\vartheta}=\dot{\vartheta}_{(\alpha)}\vec{c}_{(\alpha)0}+\dot{\varphi}_{(\alpha)}\vec{k}$<br>$\vec{\omega}_{P(\alpha)\rho}=-\frac{\dot{\vartheta}_{(\alpha)}}{\rho_{(\alpha)}\cos\vartheta_{(\alpha)}}\vec{g}_{(\alpha)\vartheta}+\dot{\varphi}_{(\alpha)}\left(\vec{g}_{(\alpha)\rho}\sin\vartheta_{(\alpha)}+\frac{1}{\rho}\vec{g}_{(\alpha)\vartheta}\cos\vartheta_{(\alpha)}\right)$ | $\vec{\omega}_{P(\alpha)\xi}=\frac{1}{\xi_{(\alpha)}\eta_{(\alpha)}\left(\eta_{(\alpha)}^{2}+\xi_{(\alpha)}^{2}\right)}\left\langle\dot{\varphi}_{(\alpha)}\eta_{(\alpha)}^{2}\xi_{(\alpha)}\vec{g}_{(\alpha)\eta}+\left(\dot{\eta}_{(\alpha)}\xi_{(\alpha)}-\dot{\xi}_{(\alpha)}\eta_{(\alpha)}\right)\vec{g}_{(\alpha)\varphi}\right\rangle$<br>$\vec{\omega}_{P(\alpha)\eta}=\frac{1}{\xi_{(\alpha)}\eta_{(\alpha)}\left(\eta_{(\alpha)}^{2}+\xi_{(\alpha)}^{2}\right)}\left\langle\left(\dot{\varphi}_{(\alpha)}\xi_{(\alpha)}^{2}\eta_{(\alpha)}\right)\vec{g}_{(\alpha)\xi}+\left(\dot{\eta}_{(\alpha)}\xi_{(\alpha)}-\dot{\xi}_{(\alpha)}\eta_{(\alpha)}\right)\vec{g}_{(\alpha)\varphi}\right\rangle$<br>$\vec{\omega}_{P(\alpha)\varphi}=\frac{1}{\xi_{(\alpha)}\eta_{(\alpha)}\left(\eta_{(\alpha)}^{2}+\xi_{(\alpha)}^{2}\right)}\left\langle\dot{\varphi}_{(\alpha)}\eta_{(\alpha)}^{2}\xi_{(\alpha)}\vec{g}_{(\alpha)\eta}+\left(-\dot{\varphi}_{(\alpha)}\xi_{(\alpha)}^{2}\eta_{(\alpha)}\right)\vec{g}_{(\alpha)\xi}\right\rangle$ |

These Christoffel's symbols, first and second kind, are expressed by corresponding covariant $g_{(\alpha)ij}(q)$, $\alpha=1,2,3,...,N$, $i,j,k=1,2,3$ or contra-variant $g_{(\alpha)}^{kl}(q)$, $\alpha=1,2,3,...,N$, $i,j,k=1,2,3$ metric tensor of corresponding position vector tangent space.

In Table 1, elements of mathematical phenomenology for three examples of the change of basic vectors of the position vector tangent space during kinetic point motion are presented in three different curvilinear coordinate systems: polar cylindrical, spherical and three-dimensional three-parabolic curvilinear coordinate systems.

For second example, changes of basic vectors $\vec{g}_{(\alpha)i}$, $\alpha=1,2,3,...,N$, $i,j,k=1,2,3$ in spherical curvilinear coordinate system of corresponding vector position tangent space with curvilinear coordinates $\rho_{(\alpha)}, \varphi_{(\alpha)}, \vartheta_{(\alpha)}$, $\alpha = 1,2,3,\ldots,N$ of kinetic point $\mathbf{N}_{(\alpha)}$, $\alpha = 1,2,3,\ldots,N$ in spherical system (see Figure 2.a*) defined as $\mathbf{N}_{(\alpha)}(\rho_{(\alpha)}, \varphi_{(\alpha)}, \vartheta_{(\alpha)})$, $\alpha = 1,2,3,\ldots,N$, and with its corresponding position defined by position vector $\vec{\rho}_{(\alpha)}(\rho_{(\alpha)}, \varphi_{(\alpha)}, \vartheta_{(\alpha)}), \alpha = 1,2,3,\ldots,N$ are determined. Using previous considerations and derived expressions, expressions of the velocity of basic vector extensions $\overset{*}{\vec{g}}_{(\alpha)i}$, $\alpha = 1,2,3,\ldots,N$, $i = 1,2,3$ and relative velocities $\varepsilon_{(\alpha)i}$ of basic vector extensions are presented in Table 1. Also, angular velocities of the rotations of each of the basic vectors $\vec{g}_{(\alpha)i}, \alpha = 1,2,3,\ldots,N, i,j,k = 1,2,3$ in spherical curvilinear coordinate system of corresponding vector position tangent space are determined from vector products (5) in the form (for detail see References [25,26] and [16,17]):

$$\vec{\omega}_{P(\alpha)\rho} = \vec{\omega}_{P(\alpha)\varphi} = \vec{\omega}_{P(\alpha)\vartheta} = \dot{\vartheta}_{(\alpha)}\vec{c}_{(\alpha)0} + \dot{\varphi}_{(\alpha)}\vec{k} = -\frac{\dot{\vartheta}_{(\alpha)}}{\rho_{(\alpha)}\cos\vartheta_{(\alpha)}}\vec{g}_{(\alpha)\vartheta} + \dot{\varphi}_{(\alpha)}\left(\vec{g}_{(\alpha)\rho}\sin\vartheta_{(\alpha)} + \frac{1}{\rho}\vec{g}_{(\alpha)\vartheta}\cos\vartheta_{(\alpha)}\right)$$
$$\alpha = 1,2,3,\ldots,N, \; i = 1,2,3. \qquad (6)$$

Each of sets of three basic vectors $\vec{g}_{(\alpha)i}, \alpha = 1,2,3,\ldots,N, i,j,k = 1,2,3$ in spherical curvilinear coordinate system of corresponding vector position tangent space is orthogonal in corresponding set of three basic vectors and kip these orthogonal angles during the kinetic point motions, and rotates with same angular velocities, but intensity of the basic vectors are changeable during the kinetic point motions.

### III.5. SOME CONCLUDING COMMENTS

In our consideration, in short presentation, the main difference between linear and nonlinear transformations, as a fundamental and the basic elements of mathematical phenomenology are pointed out. By using analysis of properties and descriptions of affine-linear and functional-nonlinear mappings, we show fundamental difference between affine (linear) and functional nonlinear - curvilinear coordinate systems with curvilinear coordinates as well as curvilinear coordinate system lines and curved coordinate system surfaces.

Formulas of coordinate transformations as well as main difference in properties of the basic vectors of position vector tangent space in affine space with linear coordinate system and corresponding basic vectors in generalized functional coordinate system with curvilinear coordinates are important for investigations of nonlinear dynamics of different dynamical systems.

Next, we show that extension of basic vectors magnitudes appear, in curvilinear coordinate systems, along kinetic point motion and expressions of velocities of the basic vector extensions are functions of kinetic point position in motion. Also, we show that rotation of the basic vectors orientations appear, in curvilinear coordinate systems, along kinetic point motion and expressions of angular velocities of basic vector rotations are also functions of kinetic point position in motion.

Obtained angular velocities, of a set of three basic vectors of position vector tangent space during kinetic point motion, permit and opens possibility to consider motion of this kinetic point, in tangent space, as a complex motion consisting of a relative motion along curvilinear coordinate line and a supported rotation motion. In three dimensional space, orthogonal and curvilinear coordinate system, it is possible to discuss Coriolis's acceleartion and inertia force of kinetic point.

Also, the obtained results, are very important, for consideration of motion of material system with multiple degrees of freedom, expressed by independant generalized curvilinear coordinates in three dimensional coordinate systems of position vector tangent spaces for each of the kinetic points and passing to multi dimensional fictive tangent space of whole material system. Then, notions about exteded tangent space, but of a mechanical system are introduced (for detail see References [25,26] and [16,17]).

## IV. ANALOGIES

The Petrović's theory [40-42] in the sixth chapter entitled: *Phenomenological analogies,* two types of analogies are listed: *Mathematical analogies and qualitative analogy*. In our opinion and present scientific knowledge is necessary to add third type of analogy: *structural analogy*.

Let's, in this chapter, to present three types of these analogies, using presentations and analysis of the elements of corresponding type analogous systems.

### IV.1. **QUALITATIVE ANALOGIES**

Starting with explanation about qualitative analogy, we take into account non-linear phenomena as it is trigger of coupled singularities containing three singular points, one no stable saddle type point and two stable centre type point with a homoclinic phase trajectory in the form of number "eight" (see References [26] and [28-38], and Figure 3.a* and c*). Each nonlinear system with cubic nonlinearity in its phase portrait contains this qualitative non-linear set named *a trigger of coupled singularities* [28,29]. This is one loop with self-cross section. Indication, that two physical disparate systems, in phase plane contain non-linear object, as it is a trigger of coupled singularities, permit to made conclusion about present qualitative analogy in local or global area of kinetic parameters in system non-linear dynamics. Then, it is possible qualitative knowledge of one of these systems behavior and properties transfer to other, taking into consideration and analysis physical, biological or other type of physical properties and system parameters.

Knowledge that system in phase plane contain trigger of couples singularities directed us to made conclusion that in this system with changes of a parameter, named as bifurcation parameter, is possible disappearance of a trigger of coupled singularities and appearance of bifurcation a stable singular point into tree singular points – trigger of coupled singularities with a no stable singular point and two stable singular points. Also, in the phase portrait is possible to indicate three (or four) types of phase trajectories (see Figure 3. a* and c*): * closed around one stable singular centre type point, correspond to periodic behavior of the system; * closed around all three  singular points –containing inside trigger of coupled singularities, correspond to double periodic behavior of the system; * closed around two stable singular centre type points, passing through no stable saddle type singular point as self-cross section point, correspond to periodic or double periodic behavior of the system; and * closed - open around all three singular points – containing inside trigger of coupled singularities, and passing through two no stable saddle type singular points, correspond to double periodic or no periodic behavior of the system.

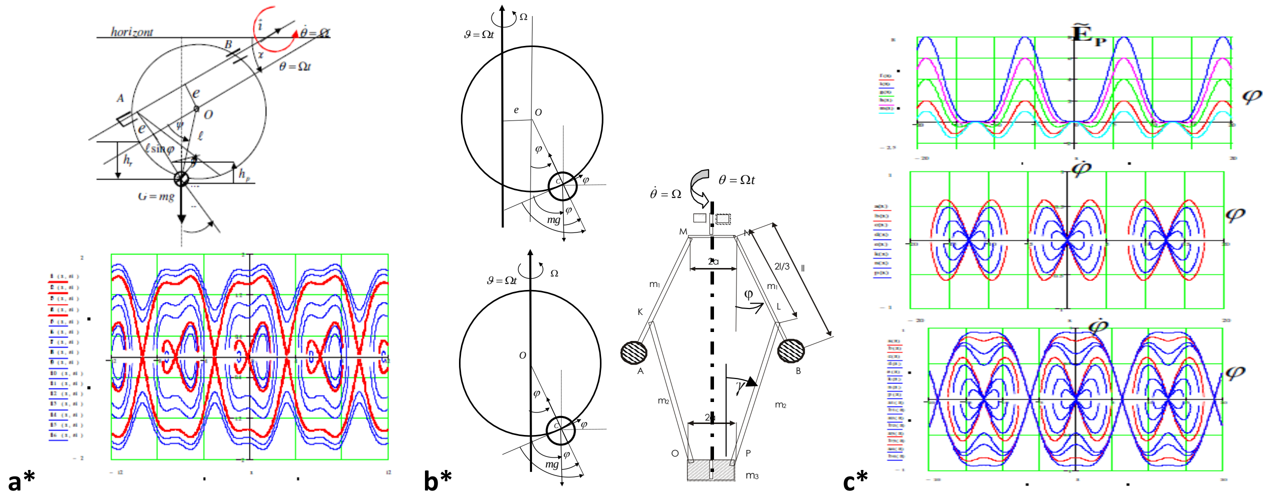


**Figure 3.** Motion of the heavy material particle along a circle, which rotates about a fixed axis: Simple model of the nonlinear dynamics. a* Phase portrait of basic model nonlinear dynamics for the vertical axis of circle rotation and for eccentricity different to zero and for b* the mechanisms of the Watt's regulator; c* Transformation (layering) of homoclinic phase trajectories in phase portrait of basic model nonlinear dynamics for the vertical axis of circle rotation and for eccentricity equal to zero.

A qualitative analogy between local phenomena of trigger of coupled singularities and bifurcation in non-linear dynamics of simple system presented in Figure 3 and in corresponding local non-linear phenomena in dynamics of a planetary geared system with debtless in gravitation field is evident. For detail see References [7, 18, 29] and system of heavy gyro-rotor with coupled rotation around no intersecting axes.[33-36,]. In the system containing trigger of coupled singularities is present a sensitive dependence of initial conditions around no stable saddle type homoclinic singular point of this trigger. If words are about non-linear dynamical system with one degree of freedom loaded by single frequency external excitation, in the system with trigger of coupled singularities is possible to conclude that it is possible to appear different types of forced regimes, and between the chaotic like and stochastic like nonlinear dynamics.

Second example of qualitative analogous phenomena in qualitative analogous systems is non-linear object named as trigger of coupled one side singular points [19]. This non-linear object appear in the system dynamics with Amontons-Coulomb's type friction and alternations of friction force directions with changes of the direction of velocity. As examples of the systems with qualitative analogous system behavior are oscillator along straight rough line and motion of a heavy mass particle along curvilinear rough line (circle, parabola, cycloid or arbitrary curvilinear line) in vertical plane (see References [38] and [37]). If curvilinear lines rotate, then by use qualitative analogy is possible to conclude that around all singular points in a trigger of coupled singularities in motion of a heavy mass particle along rotate curvilinear rough line with Coulomb's type friction appear trigger of coupled one side singular points.

## IV.2. MATHEMATICAL ANALOGIES

To explain some of possible types of mathematical analogies we use some vector models of different static or dynamic state of the systems.

### IV.2.1. MATHEMATICAL ANALOGIES BETWEEN VECTOR MODELS OF CORRESPONDING SYSTEM STATE

In dynamics of rigid body rotation is possible to use mass moment vector coupled for axis and pole introduces and defined first in 1991 by author and presented in short at IUTAM ICTAM 1992, in [4]. Important is *vector* $\vec{J}_{\vec{n}}^{(O)}$ *of mass inertia moment coupled for* *the axis of rotation passing through fixed pole (shaft bearing) (*for detail see References [5-10], [18-19], [22], [24] and [35-36]).

In signal processing, *cross-correlation* is a measure of similarity of two waveforms as a function of a time-lag applied to one of them. This is also known as a *sliding dot product* or *sliding inner-product*. It is commonly used for searching a long signal for a shorter, known feature. It has applications in pattern recognition, single particle analysis, electron homographic averaging, cryptanalysis, and neurophysiology.

For two continuous functions $f(t)$ and $g(t)$, *the cross-correlation* is defined as**:**

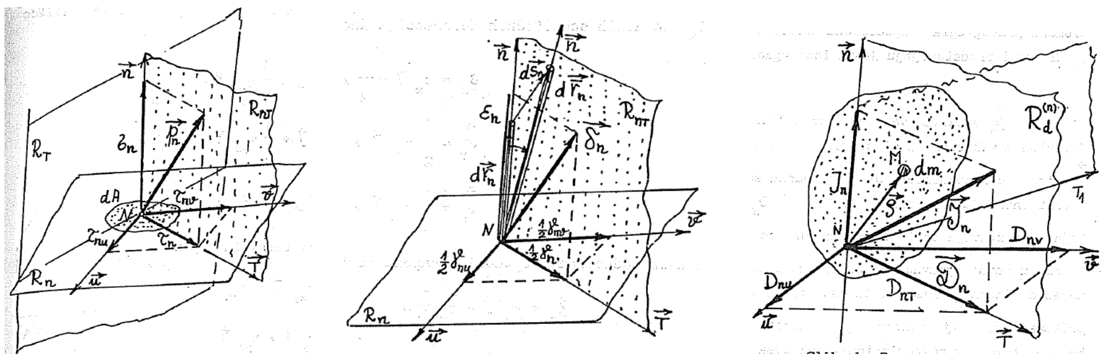
$$\mathbf{K}_{f,g}(\tau) = \left(f(t) * g(t)\right)(\tau) \stackrel{def}{=} \int_{-\infty}^{\infty} f^*(t) * g(t+\tau)dt \qquad (7)$$

where $f^*(t)$ denotes the complex conjugate function of function $f(t)$ and $t$ is the time lag. Let we have three continuous time wave processes $x(t)$, $y(t)$ and $z(t)$ in three one direction and add to each of these one of the unit vectors $\vec{n}$, $\vec{v}$ and $\vec{w}$ (or in three orthogonal directions oriented by three orthogonal unit vectors $\vec{n}$, $\vec{v}$ and $\vec{w}$), we can define a vector $\vec{\mathsf{K}}_{\vec{n}}^{(t)}(\tau)$ of the cross-correlations at moment $t$**:**

$$\vec{\mathsf{K}}_{\vec{n}}^{(t)}(\tau) = \mathbf{K}_{n,n}(\tau)\vec{n} + \mathbf{K}_{n,v}(\tau)\vec{v} + \mathbf{K}_{n,w}(\tau)\vec{w} \qquad (8)$$

**Table 2.** The mathematical analogies between vector models of stress state model, strain state model and mass inertia moment state model.

| | | | |
|---|---|---|---|
| | Vector model of stress state for point $O$ and section plane defined by unit vector $\vec{n}$, in loaded body | Vector model of strain state for point $O$ and oriented line element by unit vector $\vec{n}$ in deformed deformable body | Vector model of mass inertia moments state for point $O$ and axis oriented by unit vector $\vec{n}$ for a rigid body |
| Basic Vector | Vector of total stress $\vec{\mathbf{p}}_{\vec{n}}^{(O)}$ for section plane oriented by unit vector $\vec{n}$ through point $O$.<br>Defined by<br>$\vec{\mathbf{p}}_{\vec{n}}^{(O)} = \lim_{\Delta A_{\vec{n}}^{(O)} \to 0} \frac{\Delta \vec{\mathbf{F}}_{\vec{n}}^{(O)}}{\Delta A_{\vec{n}}^{(O)}} = \frac{d\vec{\mathbf{F}}_{\vec{n}}^{(O)}}{aA_{\vec{n}}^{(O)}}$ | Vector of total relative deformation $\vec{\delta}_{\vec{n}}^{(O)}$ of deformable body line element $dr_{\vec{n}}^{(O)}$ oriented by unit vector $\vec{n}$ through point $O$.<br>$\vec{\delta}_{\vec{n}}^{(O)} = \frac{d\vec{s}_{\vec{n},def}^{(O)}}{dr_{\vec{n}}^{(O)}}$ | Vector of mass inertia moment $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ coupled for pole $O$ and axis oriented by unit vector $\vec{n}$<br>Defined by<br>$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} \overset{def}{=} \iiint_V [\vec{\rho}, [\vec{n}, \vec{\rho}]] dm$ |
| Component vectors | $\vec{\mathbf{p}}_{\vec{n}}^{(O)} = \sigma_{\vec{n}}^{(O)} \vec{n} + \vec{\tau}_{\vec{n}}^{(O)}$<br>components: 1. normal stress component $\sigma_{\vec{n}}^{(O)} \vec{n}$ orthogonal to the plane oriented by unit vector $\vec{n}$<br>2. tangential stress component $\vec{\tau}_{\vec{n}}^{(O)}$ in section plane oriented by unit vector $\vec{n}$ through pole $O$ and in corresponding orthogonal deviational direction defined by distribution of active external surface and volume load.<br>$\vec{\tau}_{\vec{n}}^{(O)} = [\vec{n}, [\vec{\mathbf{p}}_{\vec{n}}^{(O)}, n]]$<br>$\vec{\tau}_{\vec{n}}^{(O)} = \vec{\tau}_{\vec{n},\vec{v}}^{(O)} + \vec{\tau}_{\vec{n},\vec{w}}^{(O)}$<br>$\vec{n} \perp \vec{v} \perp \vec{w}$<br>$\vec{\mathbf{p}}_{\vec{n}}^{(O)} = \sigma_{\vec{n}}^{(O)} \vec{n} + \tau_{\vec{n},\vec{v}}^{(O)} \vec{v} + \tau_{\vec{n},\vec{w}}^{(O)} \vec{w}$ | $\vec{\delta}_{\vec{n}}^{(O)} = \varepsilon_{\vec{n}}^{(O)} \vec{n} + \gamma_{\vec{n}}^{(O)} \vec{T}$<br>components: 1. normal strain component $\varepsilon_{\vec{n}}^{(O)} \vec{n}$ - relative deformation (extensions or compression) along line element oriented by unit vector $\vec{n}$<br>2. tangential strain – sliding angle component $\gamma_{\vec{n}}^{(O)} \vec{T}$ in direction orthogonal to the direction oriented by unit vector $\vec{n}$ through pole $O$ and in corresponding orthogonal deviational direction defined by distribution of of deformation caused by distribution of active external surface and volume load.<br>$\vec{n} \perp \vec{v} \perp \vec{w}$ | $\vec{\mathfrak{J}}_{\vec{n}}^{(O)} = J_{\vec{n}}^{(O)} \vec{n} + \vec{\mathfrak{D}}_{\vec{n}}^{(O)}$<br>components: 1. axial component $J_{\vec{n}}^{(O)} \vec{n}$ axial mass inertia moment for the axis oriented by unit vector $\vec{n}$ the rough pole $O$<br>2. orthogonal component $\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$ - vector of deviational mass moment for the axis oriented by unit vector $\vec{n}$ through pole $O$ and corresponding orthogonal deviational axis defined by mass distribution around primer axis.<br>$\vec{\mathfrak{D}}_{\vec{n}}^{(O)} = [\vec{n}, [\vec{\mathfrak{J}}_{\vec{n}}^{(O)}, n]]$.<br>$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} \overset{def}{=} J_{\vec{n}}^{(O)} \vec{n} + \vec{\mathfrak{D}}_{\vec{n},\vec{v}}^{(O)} + \vec{\mathfrak{D}}_{\vec{n},\vec{w}}^{(O)}$<br>$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} \overset{def}{=} J_{\vec{n}}^{(O)} \vec{n} + D_{\vec{n},\vec{v}}^{(O)} \vec{v} + D_{\vec{n},\vec{w}}^{(O)} \vec{w}$ |
| Matrix | Matrix of tensor stress state<br>$\mathbf{N}^{(\mathbf{O})} = \begin{pmatrix} \sigma_{\vec{n}}^{(O)} & \tau_{\vec{v},\vec{n}}^{(O)} & \tau_{\vec{w},\vec{n}}^{(O)} \\ \tau_{\vec{n},\vec{v}}^{(O)} & \sigma_{\vec{v}}^{(O)} & \tau_{\vec{w},\vec{v}}^{(O)} \\ \tau_{\vec{n},\vec{w}}^{(O)} & \tau_{\vec{v},\vec{w}}^{(O)} & \sigma_{\vec{w}}^{(O)} \end{pmatrix}$<br>$\vec{\mathbf{p}}_{\vec{n}}^{(O)} = \sigma_{\vec{n}}^{(O)} \vec{n} + \tau_{\vec{n},\vec{v}}^{(O)} \vec{v} + \tau_{\vec{n},\vec{w}}^{(O)} \vec{w}$<br>$\vec{\mathbf{p}}_{\vec{v}}^{(O)} = \tau_{\vec{v},\vec{n}}^{(O)} \vec{n} + \sigma_{\vec{v}}^{(O)} \vec{v} + \tau_{\vec{v},\vec{w}}^{(O)} \vec{w}$<br>$\vec{\mathbf{p}}_{\vec{w}}^{(O)} = \tau_{\vec{w},\vec{n}}^{(O)} \vec{n} + \tau_{\vec{w},\vec{v}}^{(O)} \vec{v} + \sigma_{\vec{w}}^{(O)} \vec{w}$ | Matrix of tensor stress state<br>$\mathfrak{G}^{(\mathbf{O})} = \begin{pmatrix} \varepsilon_{\vec{n}}^{(O)} & \gamma_{\vec{v},\vec{n}}^{(O)} & \gamma_{\vec{w},\vec{n}}^{(O)} \\ \gamma_{\vec{n},\vec{v}}^{(O)} & \varepsilon_{\vec{v}}^{(O)} & \gamma_{\vec{w},\vec{v}}^{(O)} \\ \gamma_{\vec{n},\vec{w}}^{(O)} & \gamma_{\vec{v},\vec{w}}^{(O)} & \varepsilon_{\vec{w}}^{(O)} \end{pmatrix}$<br>$\vec{\delta}_{\vec{n}}^{(O)} = \varepsilon_{\vec{n}}^{(O)} \vec{n} + \varepsilon_{\vec{n},\vec{v}}^{(O)} \vec{v} + \gamma_{\vec{n},\vec{w}}^{(O)} \vec{w}$<br>$\vec{\delta}_{\vec{v}}^{(O)} = \gamma_{\vec{v},\vec{n}}^{(O)} \vec{n} + \varepsilon_{\vec{v}}^{(O)} \vec{v} + \gamma_{\vec{v},\vec{w}}^{(O)} \vec{w}$<br>$\vec{\delta}_{\vec{w}}^{(O)} = \gamma_{\vec{w},\vec{n}}^{(O)} \vec{n} + \gamma_{\vec{w},\vec{v}}^{(O)} \vec{v} + \varepsilon_{\vec{w}}^{(O)} \vec{w}$ | Matrix of tensor mass inertia state<br>$\mathbf{J}^{(\mathbf{O})} = \begin{pmatrix} J_{\vec{n}}^{(O)} & D_{\vec{v},\vec{n}}^{(O)} & D_{\vec{w},\vec{n}}^{(O)} \\ D_{\vec{n},\vec{v}}^{(O)} & J_{\vec{v}}^{(O)} & D_{\vec{w},\vec{v}}^{(O)} \\ D_{\vec{n},\vec{w}}^{(O)} & D_{\vec{v},\vec{w}}^{(O)} & J_{\vec{w}}^{(O)} \end{pmatrix}$<br>$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} \overset{def}{=} J_{\vec{n}}^{(O)} \vec{n} + D_{\vec{n},\vec{v}}^{(O)} \vec{v} + D_{\vec{n},\vec{w}}^{(O)} \vec{w}$<br>$\vec{\mathfrak{J}}_{\vec{v}}^{(O)} = D_{\vec{v},\vec{n}}^{(O)} \vec{n} + J_{\vec{v}}^{(O)} \vec{v} + D_{\vec{v},\vec{w}}^{(O)} \vec{w}$<br>$\vec{\mathfrak{J}}_{\vec{w}}^{(O)} = D_{\vec{w},\vec{n}}^{(O)} \vec{n} + D_{\vec{w},\vec{v}}^{(O)} \vec{v} + J_{\vec{w}}^{(O)} \vec{w}$ |
| Main directions | Main stress directions $\vec{n}_s, s = 1,2,3$ in a point $O$ and main normal stresses $\sigma_{\vec{n}_s}^{(O)}, s = 1,2,3$ at point $O$.<br>Secular equation:<br>$\left\| \mathbf{N}^{(\mathbf{O})} - \sigma_{\vec{n}_s}^{(O)} \mathbf{I} \right\| = \mathbf{0}$<br>$\begin{vmatrix} \sigma_{\vec{n}}^{(O)} - \sigma_{\vec{n}_s}^{(O)} & \tau_{\vec{v},\vec{n}}^{(O)} & \tau_{\vec{w},\vec{n}}^{(O)} \\ \tau_{\vec{n},\vec{v}}^{(O)} & \sigma_{\vec{v}}^{(O)} - \sigma_{\vec{n}_s}^{(O)} & \tau_{\vec{w},\vec{v}}^{(O)} \\ \tau_{\vec{n},\vec{w}}^{(O)} & \tau_{\vec{v},\vec{w}}^{(O)} & \sigma_{\vec{w}}^{(O)} - \sigma_{\vec{n}_s}^{(O)} \end{vmatrix} = \mathbf{0}$ | Main strain directions $\vec{n}_s, s = 1,2,3$ in a point $O$ and main normal strains $\varepsilon_{\vec{n}_s}^{(O)}, s = 1,2,3$ at point $O$.<br>Secular equation:<br>$\left\| \mathfrak{G}^{(\mathbf{O})} - \varepsilon_{\vec{n}_s}^{(O)} \mathbf{I} \right\| = \mathbf{0}$<br>$\begin{vmatrix} \varepsilon_{\vec{n}}^{(O)} - \varepsilon_{\vec{n}_s}^{(O)} & \gamma_{\vec{v},\vec{n}}^{(O)} & \gamma_{\vec{w},\vec{n}}^{(O)} \\ \gamma_{\vec{n},\vec{v}}^{(O)} & \varepsilon_{\vec{v}}^{(O)} - \varepsilon_{\vec{n}_s}^{(O)} & \gamma_{\vec{w},\vec{v}}^{(O)} \\ \gamma_{\vec{n},\vec{w}}^{(O)} & \gamma_{\vec{v},\vec{w}}^{(O)} & \varepsilon_{\vec{w}}^{(O)} - \varepsilon_{\vec{n}_s}^{(O)} \end{vmatrix} = \mathbf{0}$ | Main inertia axes $\vec{n}_s, s = 1,2,3$ through a point $O$ and main axial mass inertia moments $J_{\vec{n}_s}^{(O)}, s = 1,2,3$ for a point $O$. Secular equation:<br>$\left\| \mathbf{J}^{(\mathbf{O})} - J_{\vec{n}_s}^{(O)} \mathbf{I} \right\| = \mathbf{0}$<br>$\begin{vmatrix} J_{\vec{n}}^{(O)} - J_{\vec{n}_s}^{(O)} & D_{\vec{v},\vec{n}}^{(O)} & D_{\vec{w},\vec{n}}^{(O)} \\ D_{\vec{n},\vec{v}}^{(O)} & J_{\vec{v}}^{(O)} - J_{\vec{n}_s}^{(O)} & D_{\vec{w},\vec{v}}^{(O)} \\ D_{\vec{n},\vec{w}}^{(O)} & D_{\vec{v},\vec{w}}^{(O)} & J_{\vec{w}}^{(O)} - J_{\vec{n}_s}^{(O)} \end{vmatrix} = \mathbf{0}$ |
| Directions of asymmetry | Extreme values of tangential stress (sheering stresses) are in surfaces oriented by unite vectors in directions under the angle of $\pi/4$ to the main stress direction $\vec{n}_s, s = 1,2,3$ in a point $O$ and equal to half value of difference between two of main normal stresses $\sigma_{\vec{n}_s}^{(O)}, s = 1,2,3$ at point $O$. | Extreme values of sliding deformation (rotation of line element) are in directions under the angle of $\pi/4$ to the main strain direction $\vec{n}_s, s = 1,2,3$ in a point $O$ and equal to half value of difference between two of main strains $\varepsilon_{\vec{n}_s}^{(O)}, s = 1,2,3$ at point $O$.<br>Between line elements in these directions are extreme values of line elements angle changes. | Extreme values of deviation component of mass inertia moment vector are in directions under the angle of $\pi/4$ to the main mass inertia moments direction $\vec{n}_s, s = 1,2,3$ in a point $O$ and equal to half value of difference between two of main axial mass inertia moments $J_{\vec{n}_s}^{(O)}, s = 1,2,3$ a point $O$. These directions are inertia astsmetry axes. |



a* b* c*

**Figure 4.** Three mathematical analogies between vector models of stress state model of a stressed body, strain state model of a deformed deformable body and mass inertia state model of a rigid body.

of which coordinates are cross correlation functions between processes $x(t)$ and $x(t), y(t)$ and $z(t)$. Autocorrelation function is in the following form:

$$\mathbf{K}_{n,n}(\tau) = (x(t) * x(t))(\tau) \stackrel{def}{=} \int_{-\infty}^{\infty} x^*(t) * x(t+\tau) dt \quad . \tag{9}$$

Matrix $\mathbf{K}$ of tensor state of cross - correlation of three collinear or orthogonal continuous time wave processes $x(t)$, $y(t)$ and $z(t)$ is in the form:

$$\mathbf{K}(\tau) = \begin{pmatrix} \mathbf{K}_{n,n}(\tau) & \mathbf{K}_{v,n}(\tau) & \mathbf{K}_{w,n}(\tau) \\ \mathbf{K}_{n,v}(\tau) & \mathbf{K}_{v,v}(\tau) & \mathbf{K}_{w,v}(\tau) \\ \mathbf{K}_{n,w}(\tau) & \mathbf{K}_{v,w}(\tau) & \mathbf{K}_{w,w}(\tau) \end{pmatrix} \tag{10}$$

Eigen numbers of previous matrix present extreme values of cross correlations and also eigen main cross-correlation directions of three orthogonal continuous time wave processes $x(t)$, $y(t)$ and $z(t)$, and, also is possible to find modal matrix for transformation of matrix $\mathbf{K}$ tensor state of cross – correlation.

Vector models of stress state and strain state in the loaded and deformed of a deformable body is known from world references in theory of elasticity (*for source see following Reference* [9] and [48]). These models are mathematically analogous. Also, is very easy to indicate mathematical analogy between vector models of cross section state of three wave processes and one of the vector models of stress state model, strain state model or mass inertia moment state model. Also, is possible to indicate analogy if these three wave processes $x(t)$, $y(t)$ and $z(t)$ are stochastic processes.

The mathematical analogies between vector models of stress state model of a stressed body, strain state model of a deformed deformable body and mass inertia moment state model of a rigid body at the point are visible in Table 2 and Figure 4.

In presented examples, elements of mathematical phenomenology–simplest mathematical analogy appear as eigen numbers of a matrix, which correspond to an set of three analogous main normal stresses, or eigen main strains, or eigen main axial mass inertia moments or main moments of stochastic processes correspond to one point.

Next elements of mathematical phenomenology-in the form of mathematical analogies, appear as eigen principal directions defined three orthogonal axes of coordinate system in which expressed matrix appears as diagonal matrix with no zero elements only at diagonal, equal to corresponding eigen numbers of a corresponding matrix in arbitrary coordinate system. These eigen principal directions are analogous to main stress directions, or main strain directions or principal inertia axes of mass inertia moments or principal moments of stochastic process in a point. Vector of principal total stress is collinear to unit vector orthogonal to the main plane, vector of main line element relative deformation is collinear with line element main direction in point, vector of mass inertia moment for main inertia direction is collinear with this direction (and rigid body is ideally balanced).

By use one mathematical analogies, vector or tensor or matrix model is possible to use for identification a qualitative analogy between stress state, strain state, mass inertia moment state or cross-correlation state of three wave orthogonal processes or the stochastic process, using an analogy between corresponding parameters which define numerous analogous physically disparate states, on the basis of mathematical or qualitative analogies.

### IV.2.2. MATHEMATICAL ANALOGIES BETWEEN MECHANICAL AND ELECTRICAL SYSTEM DYNAMICS.

Large known mathematical analogies are analogies between mechanical and electrical system dynamics, usually named as *electromechanical analogies*.

This part is necessary to formulate using knowledge from theory of mechanical and electric oscillations. Simplest explanation is by transfer signals through mechanical and electrical chains and about filter properties in transfer of signals.

***IV.2.2.1.*** In Figure 5, four systems with ideal elastic constraints and with finite number of degrees of freedom of system in mathematical analogies are presented: a* Mechanical chain system with ideal elastic springs between mass particles; b* Torsion oscillatory system in the form of a ideal elastic shaft carrying by finite number of rigid disks; c* Multi-pendulum system coupled by ideal elastic springs; d* The electric chain containing coupled finite number of electric circles (for details see References [44], [45] and [30]). These systems in Figure 5 are defined as conservative and all vibrate with constant total mechanical energy (a*, b* and c*) or total electric energy (d*) in corresponding analogous free regimes. For the case that all these systems are linear, in these mathematically analogous regimes exist corresponding numbers of eigen main modes with same numbers of eigen circular frequencies.

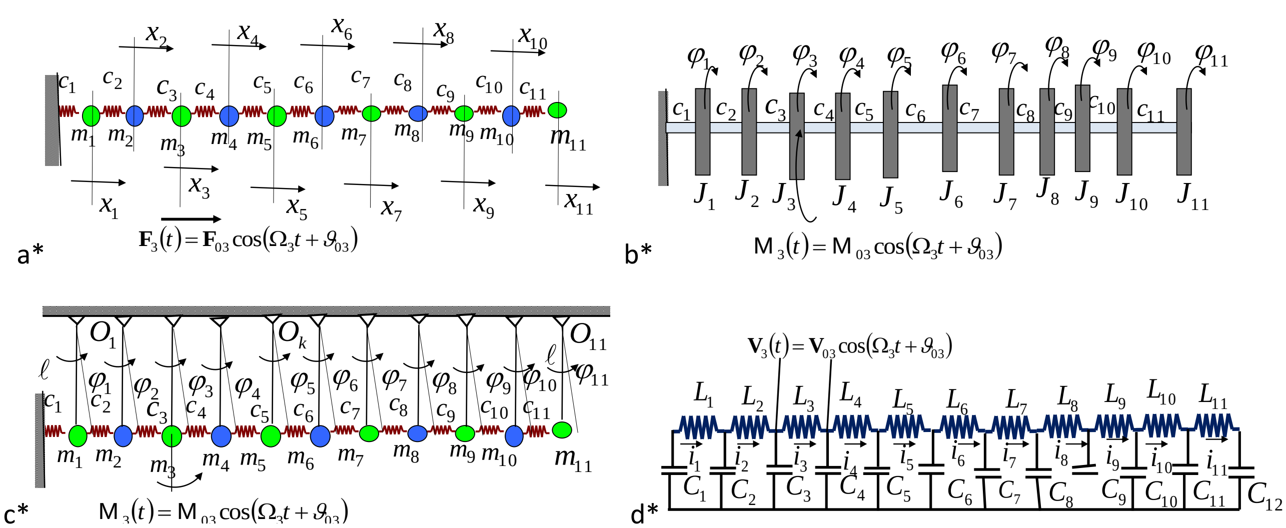
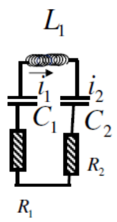
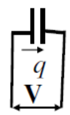
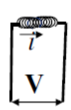
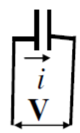
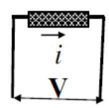
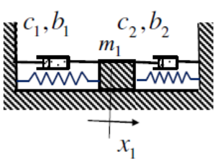
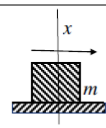
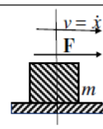
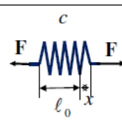
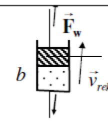



**Figure 5.** Four systems with ideal elastic constraints and with finite number of degrees of freedom of system in mathematical analogies: a* Mechanical chain system with ideal elastic springs; b* Torsion oscillatory system in the form of a ideal elastic shaft carrying by finite number of rigid disks; c* Multi-pendulum system coupled by ideal elastic springs; d* The electric chain containing coupled finite number of electric circles.

In the case that these systems are loaded by external single frequency corresponding forces of couple, in these systems appear corresponding analogous forced regimes.

All four mathematically analogous systems are defined as conservative and loaded by the external single frequency excitations. System of differential equations in matrix form of these four systems in forced regimes, presented in Figure 5, is presented in Reference [30].

**Table 3**. Mathematical analogy between kinetic and material parameters of electrical and mechanical linear oscillatory systems with one degree of freedom

| | Linear differential equations | Generalized independent coordinate of the oscillator | Coefficient of system inertia properties | Coefficient of linear elastic rigidity | Coefficient of linear dissipation of demper | External forces | Eigen circular frequency of free linear oscillator oscillations |
|---|---|---|---|---|---|---|---|
| a* | $L\ddot{q}+R\dot{q}+\frac{1}{C}q=V_0\cos\Omega t$ | $q$ electric charge (quantity of electricity) at capacitor | $L$ Coefficient of self inductivity –Inductance | $\frac{1}{C}$ Coefficient of capacitance of capacitor | $R$ is damping coefficient of electrical resistor in electrical circuit | $V_0\cos\Omega t$ External electric voltage | $\Omega_{rez}=\omega=\sqrt{\frac{1}{CL}}$ |
| b* | $m\ddot{x}+b\dot{x}+cx=F_0\cos\Omega t$ | $x$ displacement | $m$ mass | $c$ Coefficient of linear elastic rigidity of spring | $b$ is damping force component | $F_0\cos\Omega t$ force | $\Omega_{rez}=\omega=\sqrt{\frac{c}{m}}$ |
| a* |  | **C**  $q$ electric charge at capacitor | **L**  Inductance | **C**  capacitor | **R**  resistor | | |
| b* |  |  $x$ mass particle displacement |  $m$ mass particle |  spring |  damper | | |

Using analogy between forced regimes defined by analogous matrix differential equations for forced regimes in the systems presented in Figure 5, it is easier to analyze possible appearances of resonances or alternatively dynamical absorptions, which occur under corresponding relations between the eigen circular frequencies and external force frequency, as well as other system inertia or elastic or quasi-elastic or electric coefficients of the corresponding system. Detailed analysis of mathematical analogies between these system dynamics is presented by detail in new Reference [30]. In Table 3 mathematical analogy between kinetic and material parameters of electrical and mechanical linear oscillatory systems with one degree of freedom is presented.

*IV.2.2.2.* In Figure 6, two pairs of the fractional order systems with finite number of degrees of freedom of system in mathematical analogies are presented: Mechanical fractional order chain system with standard light fractional order coupling elements between mass particles and with (a*) three and (c*) finite number (eleventh) of degrees of freedom; b* The electric fractional order chain containing coupled (a*) three and (c*) finite (eleventh) number of electric circles with corresponding number of fractional order resistors. All these listed systems are no conservative and with fractional order dissipation of the corresponding system energy – total mechanical or total electrical energy fractional order dissipation along system dynamics.

*Standard light fractional order elements* [1], [21] are introduced, as coupling elements between mass particles of fractional order mechanical systems, presented in Figure 6. a* and c*. These standard light fractional order elements are defined by constitutive relation between force and elongation use a term with fractional order derivative. Also, in electrical chain fractional order system the *new standard fractional order resistors* or capacitors are introduced with constitutive relations between electric voltage and electric charge by fractional order derivatives (for detail see Reference [21], [16], [20] and [21]).

Standard light fractional order capacitive – resistive element is possible defined by following constitutive voltage-electricity charge relations [21]:

$$V(t) = -\left\{\frac{1}{C_0} q(t) + R_\alpha \mathsf{D}_t^\alpha [q(t)]\right\} \qquad (15)$$

where $V(t)$ is electrical voltage, $q(t)$ electricity charge, electric currency is $i(t) = \dot{q}(t)$ or

$q(t) = \int_0^t i(t)dt$ and $R_\alpha$ is coefficient of fractional order dissipation thermal energy in the fractional order dissipative capacitive –resistive element and $0 < \alpha \le 1$.

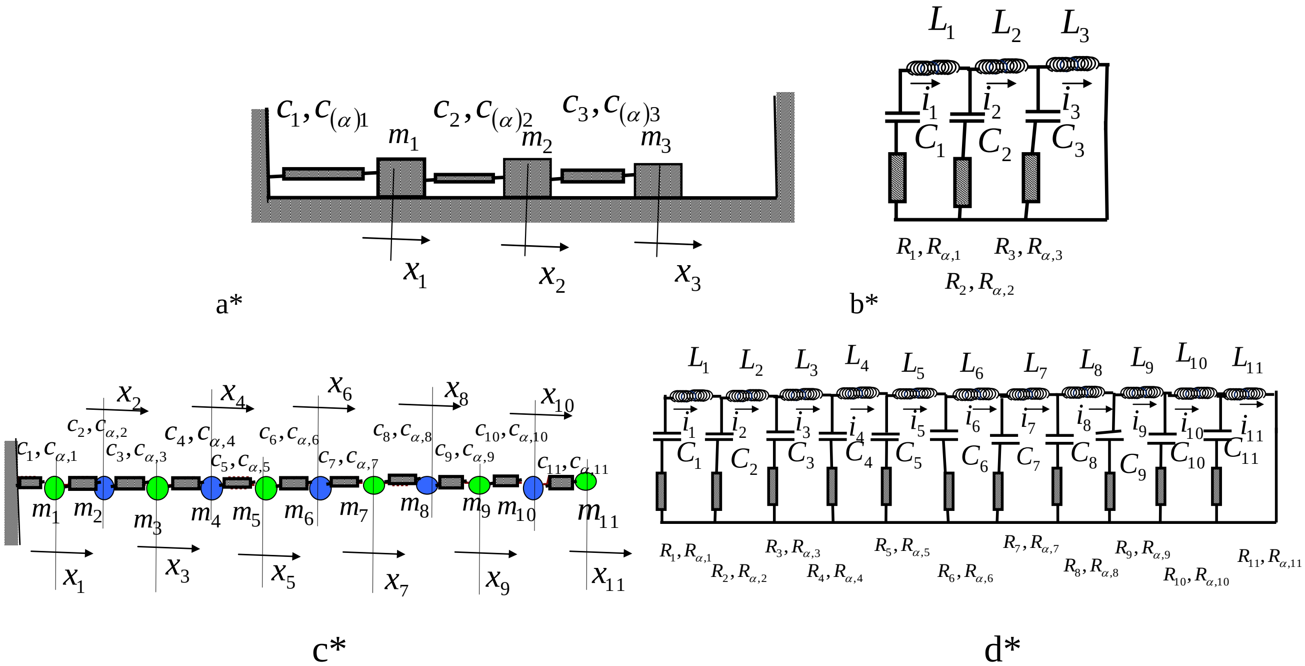



**Figure 6.** Two analogous fractional order system oscillations: fractional order mechanical chain fixed at left end and free at right end and with three a* (eleventh c*) degrees of freedom and fractional order electrical chain with three b* (d* eleventh) coupled electrical circuit and eleventh degree of freedom.

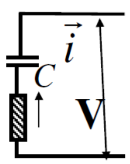
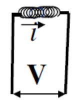
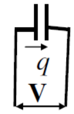
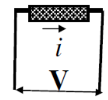
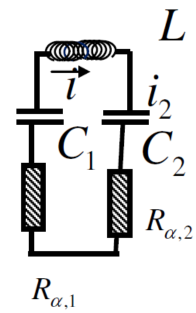
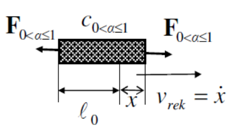
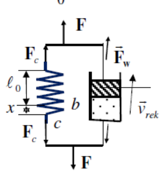
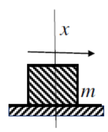
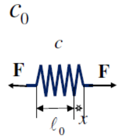
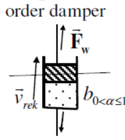
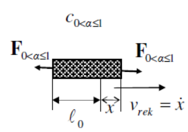
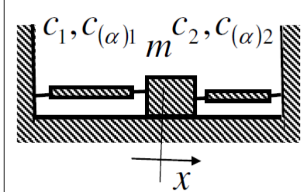
For $\alpha = 1$ voltage is $V(t) = -\left\{ \frac{1}{C_0} q(t) + R_{\alpha=1} \dot{q}(t) \right\}$. (16)

For $\alpha = 0$ voltage is:

$$V(t) = -\left\{ \frac{1}{C_0} + R_{\alpha=0} \right\} q(t) = \left\langle \frac{1}{C_0} + \frac{1}{C_{\alpha=0}} \right\rangle q(t). \quad R_{\alpha=0} = \frac{1}{C_{\alpha=0}} \quad \text{or} \quad C_{\alpha=0} = \frac{1}{R_{\alpha=0}}$$

additional capacity of capacitor.

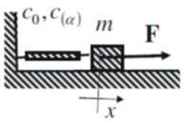
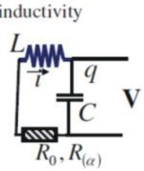
**Table 4**. Mathematical analogy between kinetic and material parameters of FO electrical and FO mechanical oscillators, for electrical resistance element and FO inertia less mechanical visco-elastic element, $0<\alpha\le 1$

| | Constitutive relation of the fractional order element | Generalized independent coordinate of the system | Coefficient of system inertia properties | Coefficient of linear elastic rigidity | Coefficients of fractional order element properties | Fractional order differential equations of the fractional order oscillatos with one degrees of freedom |
|---|---|---|---|---|---|---|
| a* | Electrical fractional order element constitutive relation electric voltage-electric charge $V(t)=-\left\{\frac{1}{C_0}q(t)+R_\alpha\mathfrak{D}_t^\alpha[q(t)]\right\}$ $0\le\alpha\le1$   $R_{0<\alpha\le1}$ $\mathbf{V}=\mathbf{V}_R+\mathbf{V}_C$ | $q$ electrical charge (quantity of electricity) at condenser | $L$ Coefficient of self inductivity **L**   | $\frac{1}{C_0}$ Coefficient of capacitor **C**   | $R_{(\alpha)}$ Coefficients of electrical resistor - element with fractional order dissipation $\mathbf{R}_{0<\alpha\le1}$   | For electrical fractional order oscillator dynamics $L\ddot{q}+R_{(\alpha)}\mathfrak{D}_t^\alpha\{q\}+\frac{1}{C_0}q=0$ $0\le\alpha\le1$   |
| b* | Mechanical fractional order element constitutive relation force- dilatation $P(t)=-\{c_0x(t)+c_\alpha\mathfrak{D}_t^\alpha[x(t)]\}$ $0\le\alpha\le1$     $\mathbf{F}=\mathbf{F}_c+\mathbf{F}_w$ | $x$ displacement of mass particle | $m$ mass   | $c_0$   | $c_{0<\alpha\le1}$ Coefficients of mechanical element fractional order dissipation= fractional order damper     | For mechanical fractional order oscillator dynamics $m\ddot{x}+c_{(\alpha)}\mathfrak{D}_t^\alpha\{x\}+cx=0$ $0\le\alpha\le1$   |

In Tables 4, we present the analogous kinetic parameters of electrical and mechanical fractional order systems.

In Table 5, the qualitative and mathematical analogies between two fractional order systems with one degree of freedom are presented. In Table 5.a* kinetic parameters of

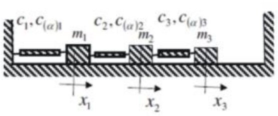
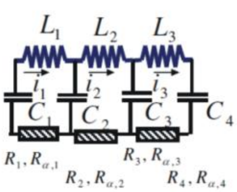
**Table 5** Qualitative and mathematical analogies between two fractional order systems with one degree of freedom: a* mechanical fractional order oscillator and b* electrical fractional order oscillator.

| | Kinetic energy<br>Coefficient of system inertia properties | Potential energy<br>Coefficient of system elasticity properties | Generalized function of fractional order energy dissipation<br>Coefficient of fractional order dissipation<br>Constitutive relation of fractional order element<br>Total system energy relation with generalized function of fractional order dissipation of system energy | Differential equation for free fractional order vibrations and characteristic numbers for eigen fractional order normal mode.<br>Particular solution modes |
|---|---|---|---|---|
| a* | $\mathbf{E}_k = \frac{1}{2} m[\dot{x}(t)]^2$<br>Kinetic energy of the mass mapticle motion<br>$m$ mass<br> | $\mathbf{E}_{p,\alpha} = \frac{1}{2} c_0 [x(t)]^2$,<br>$for \quad 0 \le \alpha < 1$<br>Potential energy of an elastic deformation of a element<br>$x$ displacement<br>$\dot{x}$ velocity<br>$c_0$ coefficient of elasticity | $\Phi_\alpha = \frac{1}{2} c_\alpha \left\langle \mathfrak{D}_t^\alpha [x(t)] \right\rangle^2$,<br>$for \quad 0 < \alpha \le 1$<br>Generalized function of fractional order energy dissipation of standard light element<br>$P(t) = -\left\{c_0 x(t) + c_\alpha \mathfrak{D}_t^\alpha [x(t)]\right\}$<br>$c_\alpha$ Coefficients of mechanical element fractional order dissipation<br>$\frac{d\mathbf{E}}{dt} = -2\Phi_s - R_\alpha \dot{x} \mathfrak{D}_t^\alpha [x]$<br>$for \quad \alpha \ne 0, 0 < \alpha < 1$ | $m\ddot{x} + c_{(\alpha)} \mathfrak{D}_t^\alpha \{x\} + cx = 0$<br>$0 \le \alpha \le 1$<br>$\omega_s^2 = \frac{c}{m}$ , $\omega_{(\alpha)}^2 = \kappa_\alpha \frac{c}{m}$<br>Particula solutions for mechanical fractional order oscillator<br>$x_{\cos}(t,\alpha) = \sum_{k=0}^{\infty} (-1)^k \omega_{(\alpha)}^{2k} t^{2k} \sum_{m=0}^{k} \binom{k}{m} \frac{\omega_{(\alpha)}^{-2m} t^{-\alpha m}}{\omega_0^{2m} \Gamma(2k+1-\alpha m)}$<br>$x_{\sin}(t,\alpha) = \sum_{k=0}^{\infty} (-1)^k \omega_{(\alpha)}^{2k} t^{2k+1} \sum_{m=0}^{k} \binom{k}{m} \frac{\omega_{(\alpha)}^{-2m} t^{-\alpha m}}{\omega_0^{2m} \Gamma(2k+2-\alpha m)}$ |
| b* | $\mathbf{E}_k = \frac{1}{2} L[\dot{q}(t)]^2$<br>Kinetic energy of the selfinguction element<br>$L$<br>Coefficient of self inductivity<br> | $\mathbf{E}_{p,\alpha} = \frac{1}{2} \frac{1}{C_0} [x(t)]^2$,<br>Potential energy of the electric capacitor<br>$q$ quantity of electricity at condenser<br>$i = \dot{q}$ electric currency<br>$q(t) = \int_0^t i(t) dt$<br>$\frac{1}{C_0}$ Coefficient of capacitor | $\Phi_\alpha = \frac{1}{2} R_\alpha \left\langle \mathfrak{D}_t^\alpha [x(t)] \right\rangle^2$,<br>$for \quad 0 < \alpha \le 1$<br>Generalized function of fractional order energy dissipation of standard fractional order electrical capacitor element<br>$V(t) = -\left\{\frac{1}{C_0} q(t) + R_\alpha \mathfrak{D}_t^\alpha [q(t)]\right\}$<br>$R_{(\alpha)}$ Coefficients of electrical resistor element fractional order dissipation<br>$\frac{d\mathbf{E}}{dt} = -2\Phi_s - R_\alpha \dot{q} \mathfrak{D}_t^\alpha [q]$<br>$for \quad \alpha \ne 0, 0 < \alpha < 1$ | $L\ddot{q} + R_{(\alpha)} \mathfrak{D}_t^\alpha \{q\} + \frac{1}{C_0} q = 0$<br>$0 \le \alpha \le 1$<br>$\omega^2 = \frac{1}{LC_0}$ , $\omega_{(\alpha)}^2 = \kappa_\alpha \frac{1}{LC_0}$<br>Particular solutions of electric currency in electruc fractional order oscillator<br>$i_{\sin}(t,\alpha) = \dot{q}_{\cos}(t,\alpha)$<br>$i_{\sin}(t,\alpha) = \sum_{k=1}^{\infty} (-1)^k \omega_\alpha^{2k} t^{2k} \sum_{m=0}^{k} \binom{k}{m} \frac{(2k-\alpha m) \omega_\alpha^{-2m} t^{-\alpha m}}{\omega_o^{2m} \Gamma(2k+1-\alpha m)}$<br>$i_{\cos}(t,\alpha) = \dot{q}_{\sin}(t,\alpha)$<br>$i_{\cos}(t,\alpha) = \sum_{k=0}^{\infty} (-1)^k \omega_\alpha^{2k} t^{2k} \sum_{m=0}^{k} \binom{k}{m} \frac{(2k+1-\alpha m) \omega_\alpha^{-2m} t^{-\alpha m}}{\omega_o^{2m} \Gamma(2k+2-\alpha m)}$ |

mechanical fractional order oscillator are presented and in Table 5.b* mathematical and qualitative analogous kinetic parameters of electrical fractional order oscillator are presented. This Table contains analogies between corresponding kinetic and potential energies, and also generalized functions of fractional order dissipation of total energy of the system fractional order dynamics (see References of theoretical basis of fractional order system dynamics with finite number of degrees of freedom in mechanical and electrical systems [21] and [16]; see also details of elements of pure mathematical phenomenology about fractional order differential equations and solution methods [3] and

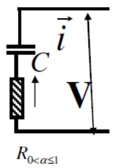
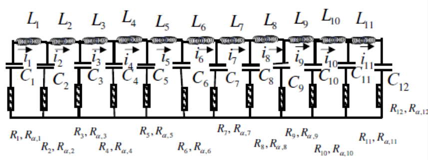
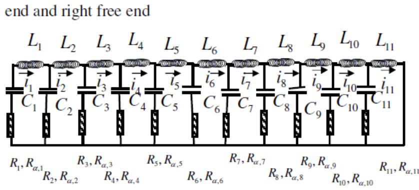
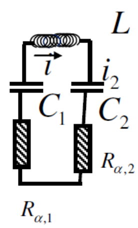
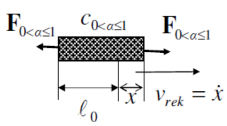
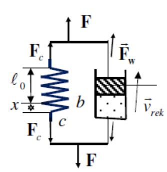
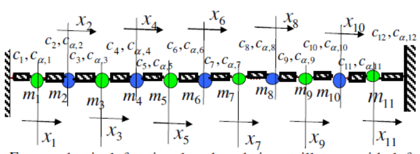
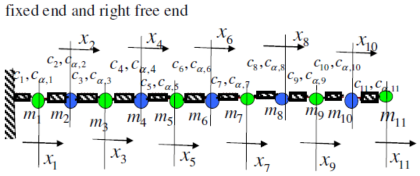
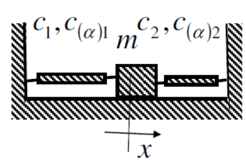
[23]; and see also elements of mathematical phenomenology in applications in engineering and bio-dynamical fractional order system dynamics [3], [20\ and [32]).

**Table 6.** Qualitative and mathematical analogous fractional order system energies: a* mechanical fractional order system and b* electrical fractional order system, with finite number of degrees of freedom.

| | Fraction ored system dynamics with finite number degrees of freedom<br>Kinetic energy | Potential energy<br>Matrix fractional order differential equation<br>Eigen main fractional order oscilators | Generalized function of fractional order energy dissipation<br>Constitutive relation of rational order element | Energy relations in fractional order system<br>Theorem of fractional order total system change |
|---|---|---|---|---|
| a* | For mass particles<br>$2\mathbf{E}_k = (\dot{x})\mathbf{A}\{\dot{x}\}$<br>$2\mathbf{E}_k = \sum_{s=1}^{s=n} \dot{\eta}_s^2$<br>$\eta_s$, $s = 1,2,3,...,n$ eigen normal coordinates (modes) of displacements<br> | For linear elastic springs<br>$2\mathbf{E}_p = (x)C\{x\}$<br>$2\mathbf{E}_p = \sum_{s=1}^{s=n} \omega_s^2 \eta_s^2$<br>$\mathbf{A}\{\ddot{x}\} + C_\alpha\{\mathfrak{D}_t^\alpha\{x\}\} + C\{x\} = \{0\}$<br>$\ddot{\xi}_s + \omega_{(\alpha)s}^2 \mathfrak{D}_t^\alpha\{\xi_s\} + \omega_s^2 \xi_s = 0$<br>$0 \le \alpha \le 1$, $s = 1,2,3,...,n$ | Power of fractional order dissipative forces in standard fractional order elements<br>$2\mathbf{P}_{\alpha\neq 0} = (\mathfrak{D}_t^\alpha\{x\})C_\alpha\{\mathfrak{D}_t^\alpha\{x\}\}$,<br>$for \quad \alpha \neq 0, 0 < \alpha < 1$<br>$2\mathbf{P}_\alpha = \sum_{s=1}^{s=n} \omega_{(\alpha),s}^2 (\mathfrak{D}_t^\alpha[\eta_s])^2$<br>$for \quad \alpha \neq 0, 0 < \alpha < 1$<br>$P(t) = -\{c_0 x(t) + c_\alpha \mathfrak{D}_t^\alpha[x(t)]\}$ | System total mechanical energy rate<br>$\frac{d\mathbf{E}}{dt} = -2\Phi - \sum_{k=1}^{k=n}\sum_{j=1}^{j=n} \dot{x}_k \frac{\partial \mathbf{P}_\alpha}{\partial(\mathfrak{D}_t^\alpha[x_j])}$, Eig<br>$for \quad \alpha \neq 0, 0 < \alpha < 1$<br>en main fractional order independent mode total mechanical energy rate<br>$\frac{d\mathbf{E}_s}{dt} = -2\Phi_s - \dot{\xi}_{ss} \frac{\partial \mathbf{P}_\alpha}{\partial(\mathfrak{D}_t^\alpha[\xi_s])}$,<br>$for \quad \alpha \neq 0, 0 < \alpha < 1, s = 1,2,3...,n$ |
| b* | For inductors<br>$2\mathbf{E}_k = (\dot{q})\mathbf{L}\{\dot{q}\}$<br>$2\mathbf{E}_k = \sum_{s=1}^{s=n} \dot{\eta}_s^2$<br>$\eta_s$, $s = 1,2,3,...,n$ eigen normal coordinates (modes) of electricity<br> | For linear capacitors<br>$2\mathbf{E}_p = (q)\mathbf{C}^*\{q\}$<br>$2\mathbf{E}_p = \sum_{s=1}^{s=n} \omega_s^2 \eta_s^2$<br>$\mathbf{A}\{\ddot{x}\} + C_\alpha\{\mathfrak{D}_t^\alpha\{x\}\} + C\{x\} = \{0\}$<br>$\ddot{\xi}_s + \omega_{(\alpha)s}^2 \mathfrak{D}_t^\alpha\{\xi_s\} + \omega_s^2 \xi_s = 0$<br>$0 \le \alpha \le 1$, $s = 1,2,3,...,n$ | Power of fractional order dissipative electrical voltage in standard fractional order resistors<br>$2\mathbf{P}_{\alpha\neq 0} = (\mathfrak{D}_t^\alpha\{q\})\mathbf{R}_\alpha\{\mathfrak{D}_t^\alpha\{q\}\}$,<br>$for \quad \alpha \neq 0, 0 < \alpha < 1$<br>$2\mathbf{P}_\alpha = \sum_{s=1}^{s=n} \omega_{(\alpha),s}^2 (\mathfrak{D}_t^\alpha[\eta_s])^2$<br>$for \quad \alpha \neq 0, 0 < \alpha < 1$<br>$V(t) = -\left\{\frac{1}{C_0} q(t) + R_\alpha \mathfrak{D}_t^\alpha[q(t)]\right\}$ | System total electrical energy rate<br>$\frac{d\mathbf{E}}{dt} = -2\Phi - \sum_{k=1}^{k=n}\sum_{j=1}^{j=n} \dot{x}_k \frac{\partial \mathbf{P}_\alpha}{\partial(\mathfrak{D}_t^\alpha[x_j])}$,<br>$for \quad \alpha \neq 0, 0 < \alpha < 1$<br>Eigen main fractional order independent mode total electrical energy rate<br>$\frac{d\mathbf{E}_s}{dt} = -2\Phi_s - \dot{\xi}_{ss} \frac{\partial \mathbf{P}_\alpha}{\partial(\mathfrak{D}_t^\alpha[\xi_s])}$,<br>$for \quad \alpha \neq 0, 0 < \alpha < 1, s = 1,2,3...,n$ |

In Table 6, the qualitative and mathematical analogies between two fractional order systems with finite number of degrees of freedom are presented. In Table 5. a* kinetic parameters of mechanical fractional order oscillator, with finite number of degrees of

**Table 7**. Analogies between matrix fractional order differential equations of FO dynamics of electrical and mechanical chain systems with finite number of loops and of dof, respectively. Their eigen FO modes, eigen characteristic numbers and corresponding constitutive relations of inertia less standard and FO electrical resistor-capacitive element and FO mechanical visco-eelastic element included in the corresponding analogous systems: $0<\alpha\le 1$. Phenomenological mapping between eigen FO modes of electrical and mechanical chains

| | Constitutive relation of the fractional order element | Matrix fractional order differential equations | Independent eigen fractional order normal oscillators | Characteristic numbers for eigen fractional order normal modes: |
|---|---|---|---|---|
| a* | In electrical fractional order system <br> $V(t) = -\left\{\frac{1}{C_0}q(t) + R_\alpha \mathfrak{D}_t^\alpha[q(t)]\right\}$ <br> $0\le\alpha\le 1$ <br>  <br> $\mathbf{V} = \mathbf{V}_R + \mathbf{V}_C$ | In electrical fractional order system <br> $\mathbf{L}\{\ddot{q}\} + \mathbf{R}_\alpha\{\mathfrak{D}_t^\alpha\{q\}\} + \mathbf{C}^*\{q\} = \{0\}$ <br> For electrical fractional order chain oscillator with fixed both ends <br>  <br> For electrical fractional order chain oscillator with left fixed end and right free end <br>  | In electrical fractional order chain system partial fractional order modes <br> $\ddot{\xi}_s + \omega_{(\alpha)s}^2 \mathfrak{D}_t^\alpha\{\xi_s\} + \omega_s^2 \xi_s = 0$ <br> $s = 1,2,3,\dots,n$ <br> , <br> $0\le\alpha\le 1$ <br>  | In electrical fractional order homogeneous chain system with both fixed ends, the characteristic numbers for independent fractional order oscillators are: <br> $\omega_s^2 = 2\frac{1}{LC_0}\left(1-\cos\frac{s\pi}{n+1}\right)$ <br> $s = 1,2,3,\dots,n$ <br> $\omega_{(\alpha)s}^2 = 2\kappa_\alpha \frac{1}{LC_0}\left(1-\cos\frac{s\pi}{n+1}\right)$ In electrical fractional order homogeneous chain system with left fixed end and right free end, the characteristic numbers for independent fractional order oscillators are: <br> $\omega_s^2 = 2\frac{1}{LC_0}\left(1-\cos\frac{(2s-1)\pi}{2n+1}\right)$ <br> $s = 1,2,3,\dots,n$ <br> $\omega_{(\alpha)s}^2 = 2\kappa_\alpha \frac{1}{LC_0}\left(1-\cos\frac{(2s-1)\pi}{2n+1}\right)$ |
| b* | In mechanical fractional order system <br> $P(t) = -\{c_0 x(t) + c_\alpha \mathfrak{D}_t^\alpha[x(t)]\}$ <br> $0\le\alpha\le 1$ <br>  <br>  <br> $\mathbf{F} = \mathbf{F}_c + \mathbf{F}_w$ | In mechanical fractional order system <br> $\mathbf{A}\{\ddot{x}\} + C_\alpha\{\mathfrak{D}_t^\alpha\{x\}\} + C\{x\} = \{0\}$ <br> For mechanical fractional order chain oscillator with fixed both ends <br>  <br> For mechanical fractional order chain oscillator with left fixed end and right free end <br>  | In mechanical fractional order chain system partial fractional order modes <br> $\ddot{\xi}_s + \omega_{(\alpha)s}^2 \mathfrak{D}_t^\alpha\{\xi_s\} + \omega_s^2 \xi_s = 0$ <br> $s = 1,2,3,\dots,n$ <br> , <br> $0\le\alpha\le 1$ <br>  | In mechanical fractional order homogeneous chain system with both fixed ends, the characteristic numbers for independent fractional order oscillators are: <br> $\omega_s^2 = 2\frac{c}{m}\left(1-\cos\frac{s\pi}{n+1}\right)$ <br> $s = 1,2,3,\dots,n$ <br> $\omega_{(\alpha)s}^2 = 2\kappa_\alpha \frac{c}{m}\left(1-\cos\frac{s\pi}{n+1}\right)$ <br> In mechanical fractional order homogeneous chain system with left fixed end and right free end, the characteristic numbers for independent fractional order oscillators are: <br> $\omega_s^2 = 2\frac{c}{m}\left(1-\cos\frac{(2s-1)\pi}{2n+1}\right)$ <br> $s = 1,2,3,\dots,n$ <br> $\omega_{(\alpha)s}^2 = 2\kappa_\alpha \frac{c}{m}\left(1-\cos\frac{(2s-1)\pi}{2n+1}\right)$ |

freedom, are presented and in Table 6.b* mathematical and qualitative analogous kinetic parameters of electrical fractional order oscillator, finite number of degrees of freedom, are presented. This Table contains analogies between corresponding kinetic and potential energies, and also generalized functions of fractional order dissipation of total energy of the system fractional order dynamics. In last right hand side column theorems of total system energy degradation are presented for all system and for each of the fractional order mode (for detail sees References [11-15 and [21]).

In Table 7, analogies between matrix fractional order differential equations of fractional order dynamics of electrical and mechanical chain systems with finite number of loops and of degree of freedom, respectively. Their eigen FO modes, eigen characteristic numbers and corresponding constitutive relations of inertia less standard and FO electrical resistor-capacitive element and FO mechanical visco-eelastic element included in the corresponding analogous systems: $0 < \alpha \leq 1$. Phenomenological mapping between eigen FO modes of electrical and mechanical chains

## VI. CONCLUDING REMARKS

On the basis of the elements of mathematical phenomenology, scientific results in World research progress, is possible to classify in the few numbers of the elements of *mathematical phenomenology* on the basis of linear or non-linear phenomena in different area of sciences and identification of general models and methods applicable, as mathematical tools, in investigation of linear or non-linear dynamics in different area of sciences.

Then, one of main research task, nowadays, is a project of reduction of models of different disparate nature systems in the basis of elements of mathematical phenomenology and corresponding qualitative, or structural, or mathematical analogies as well as by approximate phenomenological mappings around singular dynamic states. Integration knowledge on the basic elements of mathematical phenomenology will be basic knowledge kernel for future education of new university generations of students and researchers with larger scientific culture.

Task of investigation of different types of analogies is close with large knowledge about models of static and dynamics of the systems from different area of sciences. Also, for identification of analogous kinetic parameters in disparate nature system dynamics is coupled with *the capability to have a high level of intuition and intuitive recognition similar models and methods on the basis of identification of analogies.*

Author believe that this very important task, in present time, is actual for research and obtained results would lead to a systematic basis analogous models and methods, that can computerize as expert system must require each researcher. Expert base of qualitative, structural or mathematical analogous models would be very useful to every researcher, as well as to the students to develop their research capabilities and intuitive thinking.

These days appear series of journals and articles with analogous research results without knowledge, that analogous results exists long time in other area of sconces, Than, it is necessary an analysis and transfer of knowledge from one to other different area of sciences.

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